

INTRODUCTION TO CALCULUS

Single-variable calculus: limits, derivatives, integrals, the fundamental theorem, and polynomial approximation, with interactive diagrams.

OFFICIAL DOCUMENTATION

November 2026

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Limits and Continuity

How fast is a falling stone travelling *at a single instant*? Average velocity over a time interval is easy — divide distance by duration — but at one instant both quantities are zero, and $0/0$ is undefined. The same tension appears everywhere in calculus: tangent slopes, areas, infinite sums. Each is a **limit**: the value a quantity approaches as the input approaches a target.

This lesson develops the limit concept precisely enough to *prove* things about it, then builds the two structural theorems of continuous functions — the Intermediate Value Theorem and the Extreme Value Theorem — that the rest of calculus depends on.

Prerequisites. Inequalities, absolute value, and the completeness of $\mathbb{R}$ (every nonempty bounded set has a supremum). No derivative or integral machinery is used.

The Formal Definition of a Limit

Intuitively, $\lim_{x \to c} f(x) = L$ means we can make $f(x)$ arbitrarily close to L by making x sufficiently close to c. The obstacle in making this precise is circularity: "sufficiently close" must not depend on the value $f(c)$, which may not even exist. The resolution, due to Weierstrass, is to quantify the two tolerances and order them: **for every** output tolerance ϵ, **there exists** an input tolerance δ — and δ must be chosen *before* seeing any particular x.

DEFINITION 1.1.1 **Limit of a Function**

Let $f : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}$, and let c be a limit point of D (every neighborhood of c contains points of D other than c). Then

$$\lim_{x \to c} f(x) = L$$

if for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in D$:

$$0 < |x - c| < \delta \implies |f(x) - L| < \epsilon.$$

Three boundary cases deserve explicit attention:

- The condition $0 < |x - c|$ **excludes** $x = c$. The limit ignores what happens *at* the point — f need not even be defined there. This is exactly what lets tangent slopes and instantaneous velocity make sense.
- "c is a limit point" prevents vacuous truth: if c were isolated in D, some δ-neighborhood would contain no other x, and the implication would hold trivially for *any* L.
- δ may depend on ϵ (and on c), but a single δ must work for **all** x in the window simultaneously — the condition is uniform in x.

The definition is best internalized as a two-player game. Your opponent picks ϵ, shrinking the target zone $L \pm \epsilon$ on the y-axis; you must respond with δ, a window $c \pm \delta$ on the x-axis, so that *every* input from the window (except c) lands inside the target zone. You win if you have a response to *every* $\epsilon > 0$. [Spivak (4th ed.), Ch. 5 "Limits", §"Delta-Epsilon Proofs"]

Worked Example 1: a linear function

Claim. $\lim_{x \to 2}(3x - 1) = 5$.

Given $\epsilon > 0$, we must force $|f(x) - 5| < \epsilon$ by bounding $|x - 2|$. Start from the quantity we control:

$$|f(x) - 5| = |(3x - 1) - 5| = |3x - 6| = 3\,|x - 2|.$$

So $|f(x) - 5| < \epsilon$ whenever $|x - 2| < \epsilon/3$. **Choose** $\delta = \epsilon/3$. Then $0 < |x - 2| < \delta$ gives $|f(x) - 5| = 3|x - 2| < 3\delta = \epsilon$, for every x at once. The limit is proved.

Concretely: if the opponent demands $\epsilon = 0.3$, respond with $\delta = 0.1$; any x within 0.1 of 2 gives $f(x)$ within 0.3 of 5.

KNOWLEDGE CHECK

For $f(x) = 3x - 1$ at $c = 2$ with target tolerance $\epsilon = 0.30$: evaluate which proposed δ values guarantee the bound.

Answer: $\delta = 0.10$ is a valid choice satisfying the tolerance condition $|f(x) - 5| < 0.30$.; $\delta = 0.05$ is a valid choice satisfying the tolerance condition $|f(x) - 5| < 0.30$.

The algebra gives the threshold $\delta \leq \epsilon/3 = 0.10$. Both $\delta = 0.10$ and $\delta = 0.05$ succeed. Larger values like 0.30 or 3.00 fail: taking $x = 2.25$ inside the $\delta = 0.30$ window gives $|f(x) - 5| = 0.75 > 0.30$.

Worked Example 2: a nonlinear function

Claim. $\lim_{x \to 3} x^2 = 9$.

Here $|f(x) - L| = |x^2 - 9| = |x - 3|\,|x + 3|$: a *variable* factor multiplies the quantity we control. The trick is to freeze it. Impose the preliminary restriction $\delta \leq 1$: then $|x - 3| < 1$ means $2 < x < 4$, so $|x + 3| < 7$. Now

$$|x - 3| < \delta \implies |x^2 - 9| = |x - 3|\,|x + 3| < 7\,\delta.$$

Choose $\delta = \min(1, \epsilon/7)$. Then $\delta \leq 1$ activates the bound $|x + 3| < 7$, and $7\delta \leq \epsilon$ finishes: $|x^2 - 9| < \epsilon$ for every x with $0 < |x - 3| < \delta$.

Compare the two proofs. In Example 1 the response was proportional ($\delta = \epsilon/3$); in Example 2 it is not, because the sensitivity of x^2 to changes in x varies *along* the window. The further c is from 0, the smaller the working δ for the same ϵ — the definition captures local sensitivity, not a global property of the formula.

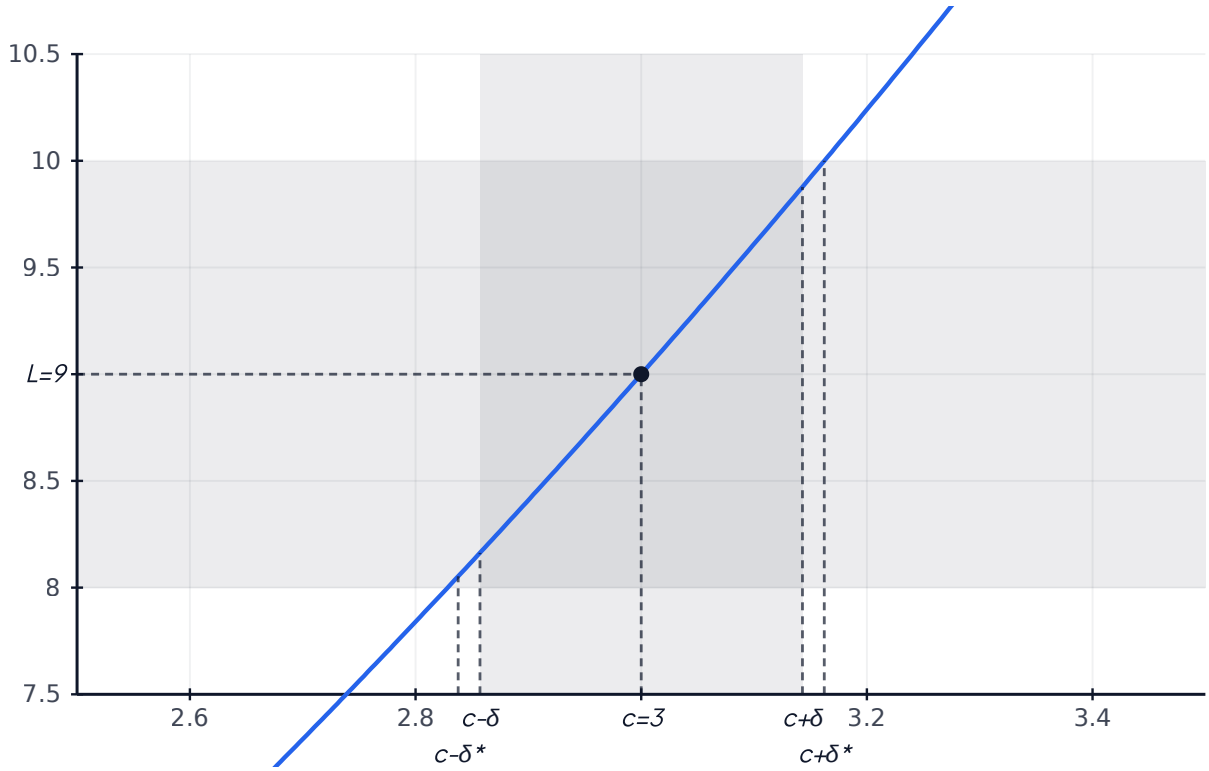


Figure 1.1.1: Epsilon-delta verification for $\lim_{x \to 3} x^2 = 9$ with $\varepsilon = 1$ and preliminary bound $\delta = \frac{1}{7}$.

KNOWLEDGE CHECK

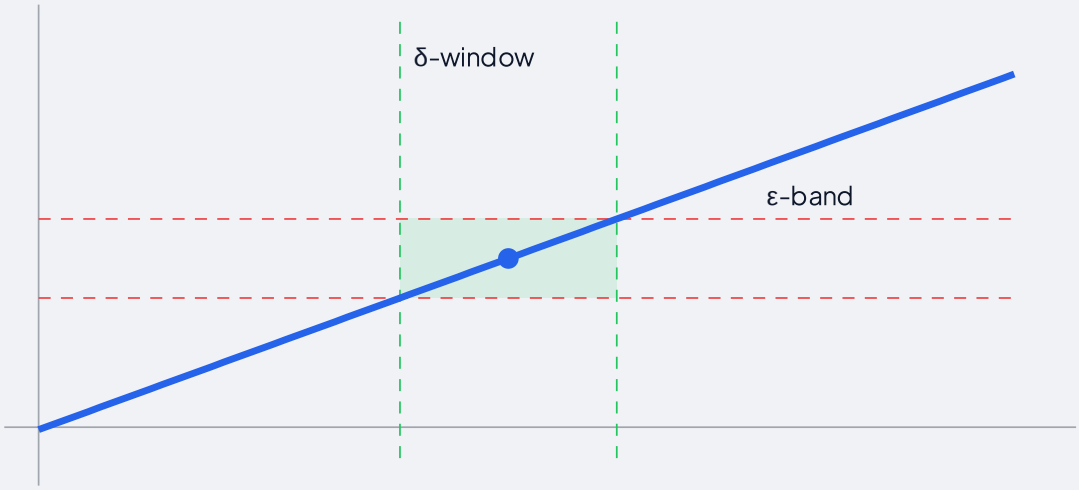
In proving $\lim_{x \to 3} x^2 = 9$, evaluate each statement concerning the preliminary bound $\delta \leq 1$:

Answer: It restricts x to the open interval $(2, 4)$, bounding $|x + 3| < 7$.; It decouples the local scale so $\delta = \min(1, \epsilon/7)$ controls error.

The factor $|x + 3|$ depends on x, so an estimate like $|x - 3|\,|x + 3| < \delta\,\epsilon$ is not valid — δ was not chosen relative to $|x + 3|$. Restricting $|x - 3| < 1$ confines x to $(2, 4)$, where $|x + 3| < 7$ is a fixed constant. Excluding $x = c$ is already built into the limit definition ($0 < |x - c|$), so no extra restriction is needed for that.

The geometry of this game, for the linear example: the horizontal ϵ-band around $L = 5$ pulls back through the line to a vertical δ-window around $c = 2$, and $\delta = \epsilon/3$ is the exact threshold.

MOVE THE ε-BAND



For f(x)=3x−1 at c=2, the δ-window pulls back the ε-band around L=5.

Fundamental Limit Theorems

Computing limits directly from the ϵ-δ definition for every new function would be unusable. The algebra of limits lets us assemble complicated limits from simple ones, *provided* the pieces exist and the operations are defined.

THEOREM 1.1.2 **Algebra of Limits**

Builds on: Definition 1.1.1

Suppose $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, and c is a limit point of the relevant domains. Then:

1. $\lim_{x \to c} (f + g)(x) = L + M$;
2. $\lim_{x \to c} (f \cdot g)(x) = L \cdot M$;
3. if additionally $M \neq 0$, then $\lim_{x \to c} (f/g)(x) = L/M$.

We prove the sum rule; the product and quotient rules follow the same pattern with an extra preliminary bound (Problem 1.P.1).

Let $\epsilon > 0$.

1. Since $\lim_{x \to c} f(x) = L$, there exists $\delta_1 > 0$ such that $0 < |x - c| < \delta_1 \implies |f(x) - L| < \epsilon/2$.
2. Since $\lim_{x \to c} g(x) = M$, there exists $\delta_2 > 0$ such that $0 < |x - c| < \delta_2 \implies |g(x) - M| < \epsilon/2$.
3. Set $\delta = \min(\delta_1, \delta_2)$. For $0 < |x - c| < \delta$ both bounds hold, so the triangle inequality gives:

$$|(f + g)(x) - (L + M)| \leq |f(x) - L| + |g(x) - M| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus $\lim_{x \to c}(f + g)(x) = L + M$.

∎

The **quotient rule's hypothesis $M \neq 0$ is essential and is the most common source of wrong limits**. The classic $\lim_{x \to 0} \frac{\sin x}{x}$ cannot be evaluated by the quotient rule: $\lim_{x \to 0} \sin x = 0$ and $\lim_{x \to 0} x = 0$, so rule 3 does not apply. The correct value is 1, but it must be proved by squeezing, not by algebra [Spivak (4th ed.), Ch. 15, §"The Derivatives of Trigonometric Functions"]. The algebra of limits never resolves a $0/0$ form; that is precisely the case calculus exists to handle.

KNOWLEDGE CHECK

A proposed calculation posits $\lim_{x\to 0} \frac{\sin x}{x} = \frac{\lim_{x\to 0} \sin x}{\lim_{x\to 0} x} = \frac{0}{0}$ and concludes the limit does not exist. Evaluate each claim about this calculation:

Answer: The quotient rule cannot be applied because denominator limit is zero.; An indeterminate form $0/0$ does not imply that the limit diverges.

Both numerator and denominator limits are correctly computed as 0; the mistake is applying the quotient rule outside its hypothesis $M \neq 0$. The symbol $0/0$ is not a limit value — it is a signal that the rule is silent. (The actual limit is 1, proved by a geometric squeeze.)

THEOREM 1.1.3 **Squeeze Theorem**

Builds on: Definition 1.1.1

Let I be an open interval containing c, except possibly at c itself. Let f, g, h be functions defined on I. If for all $x \in I \setminus \{c\}$,

$$g(x) \leq f(x) \leq h(x)$$

and $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$, then:

$$\lim_{x\to c} f(x) = L.$$

Let $\epsilon > 0$ be given.

1. Since $\lim_{x \to c} g(x) = L$, there exists $\delta_1 > 0$ such that for all $0 < |x - c| < \delta_1$:

$$|g(x) - L| < \epsilon \iff L - \epsilon < g(x) < L + \epsilon.$$

2. Since $\lim_{x \to c} h(x) = L$, there exists $\delta_2 > 0$ such that for all $0 < |x - c| < \delta_2$:

$$|h(x) - L| < \epsilon \iff L - \epsilon < h(x) < L + \epsilon.$$

3. Choose $\delta = \min(\delta_1, \delta_2) > 0$. For all x satisfying $0 < |x - c| < \delta$, both inequalities hold simultaneously. Combining with the sandwiching hypothesis $g(x) \leq f(x) \leq h(x)$:

$$L - \epsilon < g(x) \leq f(x) \leq h(x) < L + \epsilon.$$

Extracting the outer bounds yields:

$$L - \epsilon < f(x) < L + \epsilon \iff |f(x) - L| < \epsilon.$$

This establishes that $\lim_{x \to c} f(x) = L$.

∎

The Squeeze Theorem is the standard tool for functions that *oscillate too fast to control directly*. A classic application: $\lim_{x \to 0} x \sin(1/x) = 0$ (and similarly $\lim_{x \to 0} x^2 \sin(1/x) = 0$). Since $-1 \leq \sin(1/x) \leq 1$ for all $x \neq 0$, multiplying by $|x| > 0$ yields $-|x| \leq x \sin(1/x) \leq |x|$ (or $-x^2 \leq x^2 \sin(1/x) \leq x^2$). Both bounds tend to 0 as $x \to 0$, so the middle function does too — *regardless of how fast it oscillates inside the envelope*. The theorem transfers existence, not rates: it says nothing about *how fast* f approaches L.

Contrast with a function the squeeze *cannot* rescue: $\sin(1/x)$ itself has no limit at 0 — no envelope tighter than $[-1, 1]$ is available, and the values cycle through all of $[-1, 1]$ in every neighborhood of 0 (see the counterexample below).

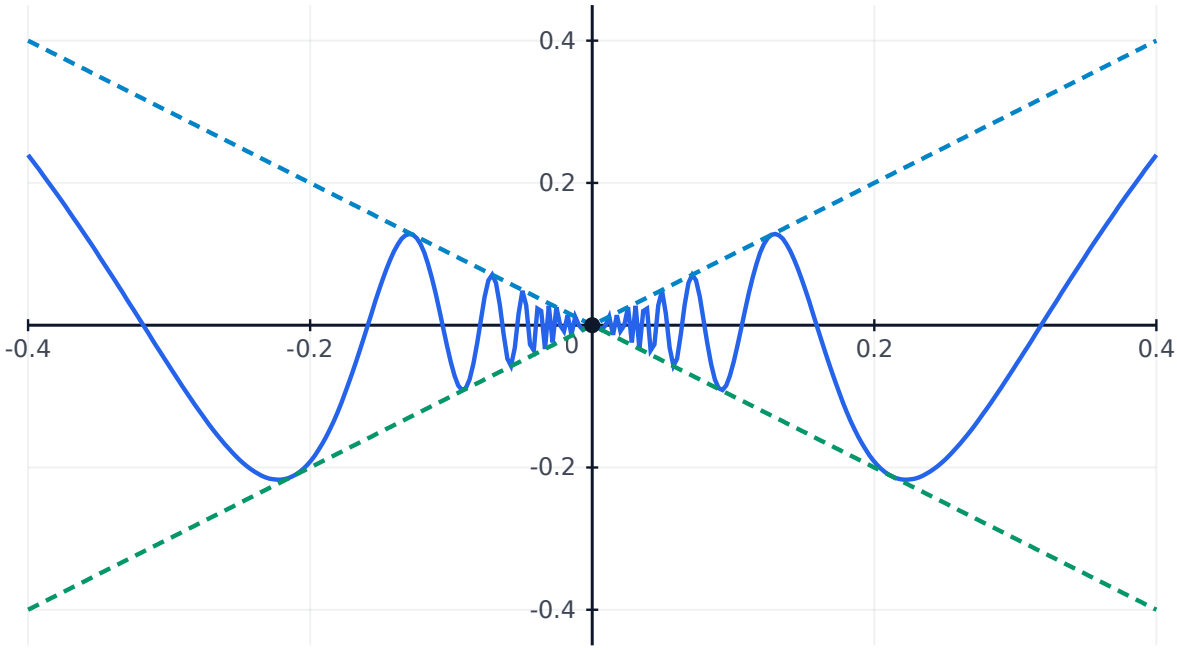


Figure 1.1.2: The squeeze theorem: $-|x| \le x\sin(1/x) \le |x|$, trapping the oscillating function at the origin.

Continuity

Continuity formalises "no gaps or jumps" — the input–output contract that makes the two structural theorems below possible.

DEFINITION 1.1.4 **Continuity**

A function $f : D \to \mathbb{R}$ is continuous at a point $c \in D$ if:

$$\lim_{x \to c} f(x) = f(c).$$

The function is continuous on an interval $I \subseteq D$ if it is continuous at every $c \in I$.

Unpacking the definition: continuity at c requires (i) $f(c)$ is defined, (ii) $\lim_{x \to c} f(x)$ exists, and (iii) the two are equal. Each requirement fails in a different, characteristic way, giving a taxonomy of discontinuities:

Type	Definition failure	Canonical example	Continuous extension possible?
Removable	(i) or (iii): limit exists but $f(c)$ is undefined or differs	$f(x) = (x^2 - 1)/(x - 1)$ at $c = 1$ (limit $= 2$) or $\sin(x)/x$ at $c = 0$ (limit $= 1$)	Yes: (re)define $f(c) = \lim_{x \to c} f(x)$
Jump	(ii): one-sided limits exist but differ	$f(x) = \operatorname{sgn}(x)$ at $c = 0$: limits ∓ 1	No
Essential	(ii): limit does not exist, even one-sided	$f(x) = \sin(1/x)$ at $c = 0$	No

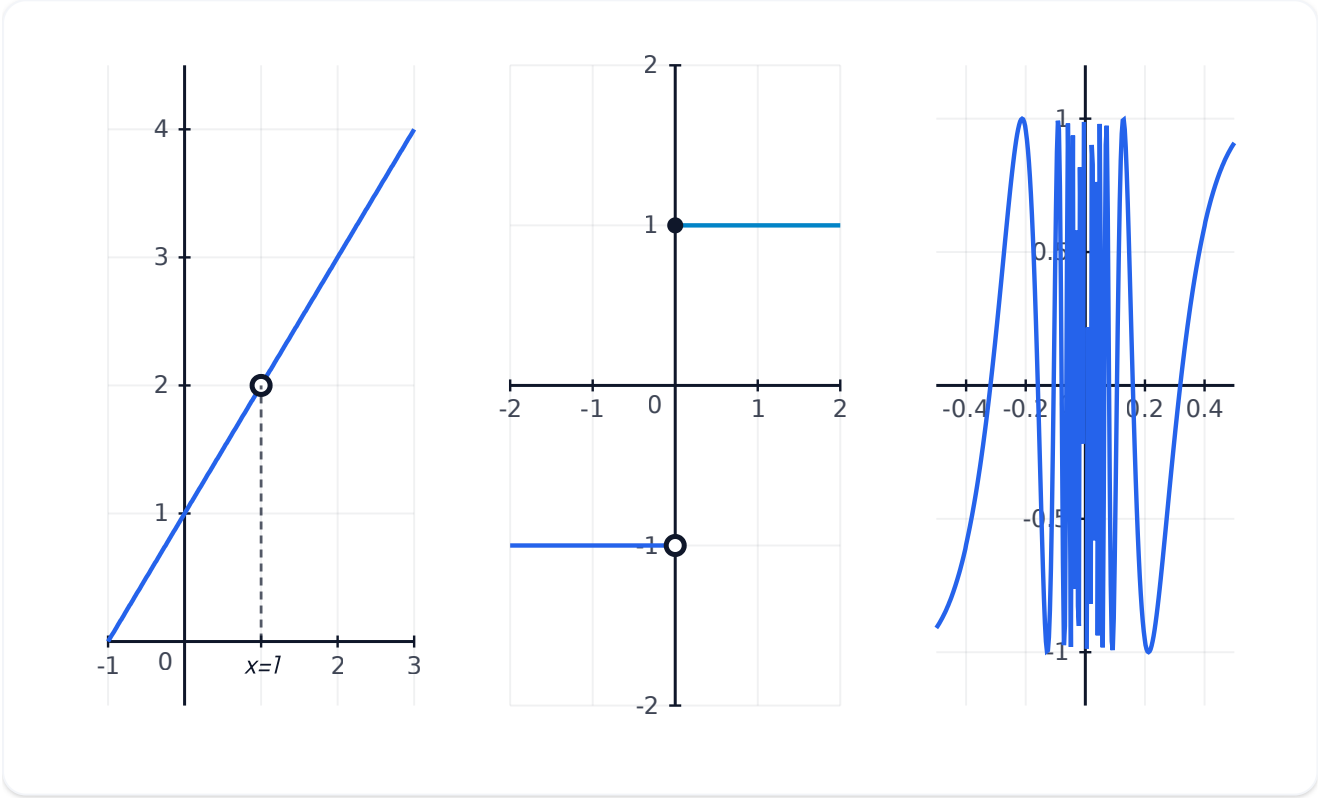


Figure 1.1.3: Three types of discontinuity: removable (hole at $(1, 2)$), jump (finite gap at $x = 0$), and essential (infinite oscillation as $x \to 0$).

Sums and products of continuous functions are continuous (transfer the Algebra of Limits termwise); **quotients are continuous only where the denominator is nonzero** — the quotient-rule hypothesis again.

Counterexample: essential discontinuity

Consider $f(x) = \sin(1/x)$ for $x \neq 0$. The limit as $x \to 0$ does not exist, so no definition of $f(0)$ can make f continuous — the discontinuity is essential.

KNOWLEDGE CHECK

Consider $f(x) = \sin(1/x)$ on $\mathbb{R} \setminus \{0\}$. Evaluate each claim regarding its limiting behavior as $x \to 0$:

Answer: Sequences $x_n = 1/(\pi/2 + 2\pi n)$ and $y_n = 1/(3\pi/2 + 2\pi n)$ converge to different limits.; Every punctured neighborhood $(-\delta, \delta) \setminus \{0\}$ contains values spanning the full range $[-1, 1]$.

Although $f(0)$ is undefined, a limit could still exist — the limit definition excludes $x = c$. It does not: every neighborhood of 0 contains points where $\sin(1/x) = 1$ (namely $x = 1/(\pi/2 + 2k\pi)$) and points where it equals -1 (namely $x = 1/(3\pi/2 + 2k\pi)$). Any candidate L fails ϵ-control for $\epsilon = 1/2$: no δ-window avoids values a full unit apart.

Compare with $x^2 \sin(1/x)$: multiplying the wild factor by x^2 tames it enough for the Squeeze Theorem, and the extension $F(0) = 0$ *is* continuous. The boundary between "tameable" and "untameable" oscillation is exactly what a squeeze envelope can control — a distinction that returns when derivatives are defined.

The Intermediate Value Theorem (IVT)

Continuous functions on closed intervals cannot skip values. This is the theorem that guarantees roots exist before we compute them.

THEOREM 1.1.5 **Intermediate Value Theorem (IVT)**

Builds on: Definition 1.1.4

If $f : [a, b] \to \mathbb{R}$ is continuous and u is any number between $f(a)$ and $f(b)$, then there exists at least one $c \in (a, b)$ such that:

$$f(c) = u.$$

The proof [Spivak (4th ed.), Ch. 8 "Least Upper Bound", Theorem 1; Rudin, *PMA* (3rd ed.), Ch. 4, Thm 4.23] constructs $c = \sup\{x \in [a,b] : f(x) < u\}$ — a set that is nonempty and bounded by completeness — and shows continuity forces $f(c) = u$ exactly. Completeness is not decoration: on $\mathbb{Q}$ the theorem fails ($f(x) = x^2$ is continuous on $[1,2] \cap \mathbb{Q}$, takes endpoint values 1 and 4, but no rational has square 2).

Both hypotheses are load-bearing:

- *Discontinuity:* $f(x) = \operatorname{sgn}(x)$ on $[-1,1]$ jumps from -1 to 1 and never takes the value $u = 0$ between its endpoint values.
- *Endpoint values:* the theorem cannot even be *stated* without $f(a), f(b)$ — this is where "closed interval" enters.

KNOWLEDGE CHECK

A continuous function on $[-2,2]$ satisfies $f(-2) = 3$ and $f(2) = -1$. Evaluate each statement:

Answer: By the Intermediate Value Theorem, there exists at least one root $c \in (-2,2)$ where $f(c) = 0$.; The intermediate value property guarantees that every real value between -1 and 3 is attained.

The IVT requires only that u lie between $f(a)$ and $f(b)$ — in either order. Wiggling cannot dodge the value: to get from 3 to -1 the function must pass through 0 at least once. A constant function with $f(-2) = 3, f(2) = -1$ cannot exist.

The IVT also gives a *numerical* algorithm. If f is continuous and $f(a), f(b)$ have opposite signs, the sign change survives on exactly one half of $[a,b]$ after bisecting at $m = (a+b)/2$, and that half must contain a root. Repeating halves the error each step: after n bisections the root is pinned to within $(b-a)/2^n$, so a tolerance τ is guaranteed by any $n \geq \log_2((b-a)/\tau)$. [Burden & Faires, *Numerical Analysis* (9th ed.), §2.1 "The Bisection Method"]

INTERACTIVE LAB

```
def bisection_method(f, a, b, tol=1e-5):
  if f(a) * f(b) >= 0:
      raise ValueError("IVT condition not met: f(a) and f(b) must have opposite signs.")

  while (b - a) / 2.0 > tol:
      midpoint = (a + b) / 2.0
      if f(midpoint) == 0:
          return midpoint
      elif f(a) * f(midpoint) < 0:
          b = midpoint
      else:
          a = midpoint
  return (a + b) / 2.0

# Find the root of f(x) = x^3 - x - 2 on [1, 2].
# Check: f(1) = -2 < 0 < f(2) = 4, so the IVT guarantees a root in (1, 2).
f = lambda x: x**3 - x - 2
root = bisection_method(f, 1, 2)
print(f"Calculated Root: {root:.6f}")
print(f"f(root) = {f(root):.6f}")
```

Verify the error bound. The loop exits when $(b-a)/2 \le 10^{-5}$; since the interval starts with length 1 and each pass halves it, after n passes the length is 2^{-n}, and $2^{-17} < 10^{-5}$. So the printed root is within 10^{-5} of a true root of $x^3 - x - 2$ — a claim about the *real* root, justified by the IVT plus halving arithmetic, not by inspecting the output.

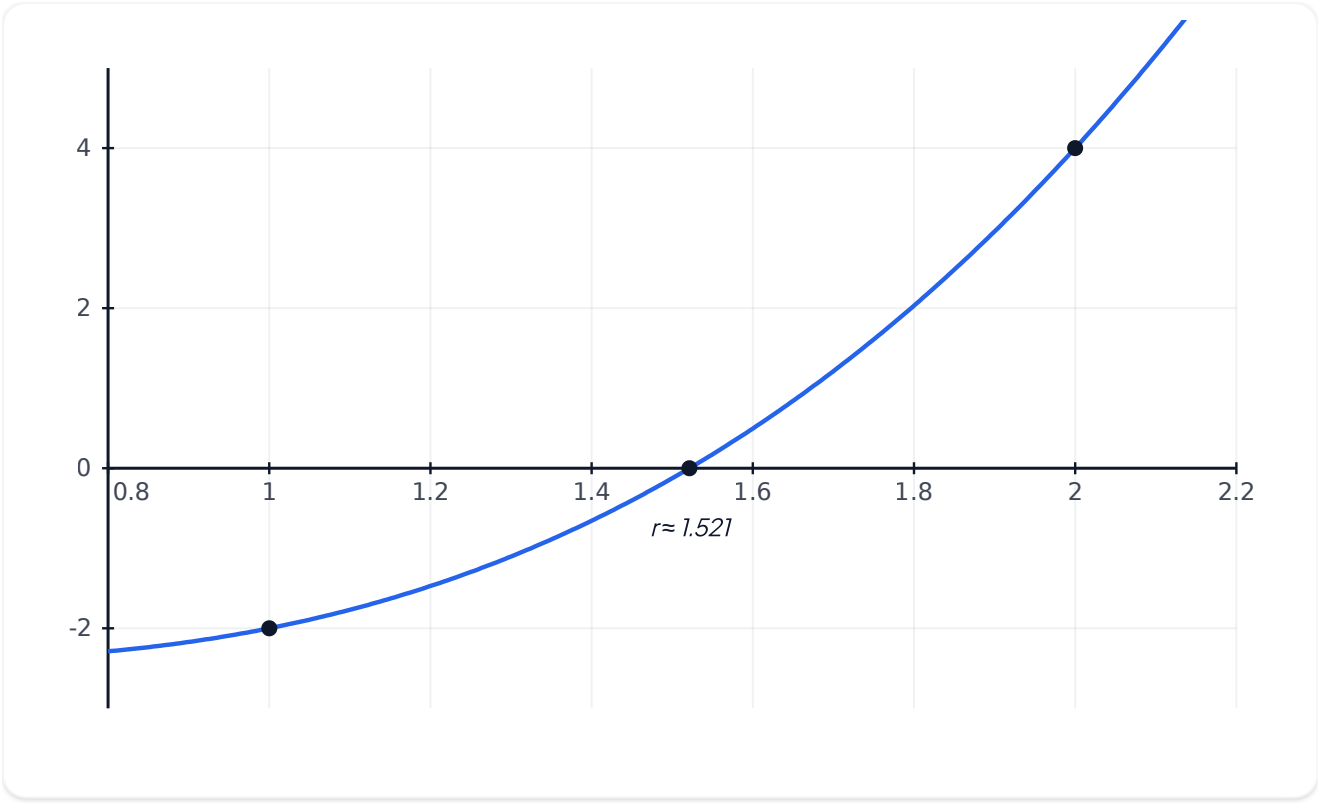


Figure 1.1.4: The Intermediate Value Theorem guarantees a root $r \approx 1.5214$ in $[1, 2]$ because $f(1) = -2 < 0 < 4 = f(2)$.

The Extreme Value Theorem (EVT)

Continuity on a closed, bounded interval does more than prevent skipped values — it keeps f bounded and forces the bounds to be *attained*. [Spivak (4th ed.), Ch. 7 "Three Hard Theorems", Thms 1–2; Rudin, *PMA* (3rd ed.), Ch. 4, Thms 4.16–4.17]

THEOREM 1.1.6 **Extreme Value Theorem (EVT)**

Builds on: Definition 1.1.4

If $f : [a, b] \to \mathbb{R}$ is continuous, then f attains a maximum and a minimum: there exist points $x_{\min}, x_{\max} \in [a, b]$ such that

$$f(x_{\min}) \leq f(x) \leq f(x_{\max}) \quad \text{for all } x \in [a, b].$$

The theorem has three hypotheses — continuity, closedness, boundedness of the interval — and each is falsifiable by a concrete counterexample:

- **Open or half-open interval fails.** $f(x) = x$ on $(0, 1)$ is continuous and bounded ($0 < f < 1$), but no point attains 0 or 1; even more dramatically, $f(x) = 1/x$ on $(0, 1]$ is continuous on its domain but blows up as $x \to 0^+$, so it is not even bounded above (shown below).
- **Unbounded interval fails.** $f(x) = x$ on $[0, \infty)$ is continuous but unbounded above.
- **Discontinuity fails.** Define $f(x) = x$ on the open part $(0, 1)$ and $f(0) = f(1) = -1$. This f is bounded on a closed interval, but its values approach 1 from below without ever attaining it: the jump at the endpoints dodges the supremum, so no maximum exists.

Like the IVT, the EVT is a completeness theorem: its proof [Spivak (4th ed.), Ch. 8; Rudin, *PMA*, Ch. 4, Thm 4.16] shows the set $f([a, b])$ is bounded and that $\sup f([a, b])$ is a value of f — both steps requiring least upper bounds.

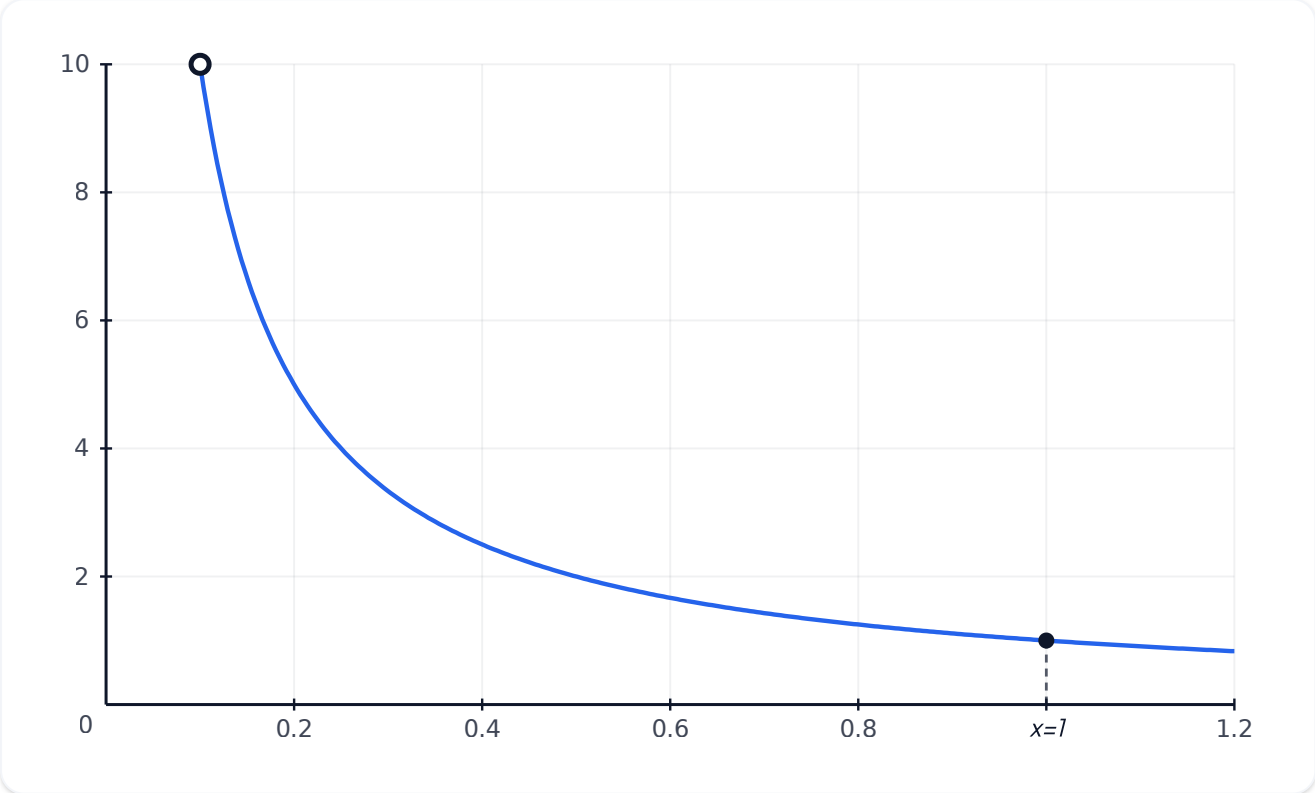


Figure 1.1.5: Failure of the Extreme Value Theorem on $(0, 1]$: $f(x) = \frac{1}{x}$ is continuous but fails to attain a maximum as $x \to 0^+$.

KNOWLEDGE CHECK

For $f(x) = x^2$ on the open interval $(-1, 1)$, evaluate each statement:

Answer: The function attains an absolute minimum value 0 at the interior point $x = 0$.; The function possesses a least upper bound $\sup f = 1$ that is never attained as a maximum.

The minimum 0 is attained at $x = 0$, inside the interval. The values $f(x)$ approach but never reach 1, because $x = \pm 1$ are excluded — $\sup f((-1, 1)) = 1$ is not a value of f. The EVT is silent here because its hypothesis 'closed interval' fails; note that EVT failing for the maximum does not imply it fails for the minimum.

The IVT and EVT are complementary guarantees: the IVT says the image $f([a, b])$ has *no gaps between attained values*, the EVT says the image is *bounded and contains its own endpoints*. Together they make the image of a compact interval a closed, bounded — and, for the derivative-theorems to come, decisive — object.

SECTION EXERCISES

Deep Practice

Solve these problems to test and consolidate your understanding of epsilon-delta limits, theorem hypotheses, and continuity contracts. Complete worked solutions and verification criteria are available for each problem.

Exercise 1.1.1: Linear Epsilon-Delta Verification

Verification application

Prove directly from the formal definition of a limit (Definition 1.1.1) that $\lim_{x \to 1} (2x + 3) = 5$.

Compute $|f(x) - 5|$ in terms of $|x - 1|$ first; the constant 3 is eliminated before choosing δ.

Given $\epsilon > 0$, choose $\delta = \epsilon/2$. Then $0 < |x - 1| < \delta$ gives

$$|(2x + 3) - 5| = |2x - 2| = 2\,|x - 1| < 2\delta = \epsilon.$$

Verification criteria. A complete derivation requires: (a) an explicit δ depending on ϵ; (b) the implication stated for *all* x with $0 < |x - 1| < \delta$, not just one; (c) the chain of equalities ending in a quantity $< \epsilon$. A response that verifies the limit only numerically (e.g., plugging in $x = 1.01$) is invalid.

Exercise 1.1.2: Quadratic Sensitivity and Preliminary Bound

Construction application

Prove from Definition 1.1.1 that $\lim_{x \to 2} x^2 = 4$, giving δ explicitly in terms of ϵ.

Factor $|x^2 - 4| = |x - 2|\,|x + 2|$ and freeze the variable factor with a preliminary bound $\delta \leq 1$, as in Worked Example 2.

Given $\epsilon > 0$, choose $\delta = \min(1, \epsilon/5)$. If $|x - 2| < \delta \leq 1$, then $1 < x < 3$, so $|x + 2| < 5$, and

$$|x^2 - 4| = |x - 2|\,|x + 2| < 5\,\delta \leq 5 \cdot \frac{\epsilon}{5} = \epsilon.$$

Verification criteria. Any correct δ (e.g., $\min(1, \epsilon/5)$) is valid; a response of $\delta = \epsilon/5$ *without* the preliminary bound is a **defect**, because $|x + 2| < 5$ was used without justification — that inequality needs $|x - 2| < 1$. A response of $\delta = \epsilon/4$ is falsifiable: with $\epsilon = 0.4$, take $x = 2.1$, then $|x^2 - 4| = 0.41 > 0.4$.

Exercise 1.1.3: Oscillating Squeeze vs Removable Discontinuity

Description transfer

Define $f(x) = x \sin(1/x)$ for $x \neq 0$. Show that $\lim_{x \to 0} f(x) = 0$, and state where f can be extended continuously. Then contrast: does $\sin(1/x)$ have a limit at 0?

Bound $|x \sin(1/x)|$ by a quantity that ignores the sine, then apply the Squeeze Theorem (Theorem 1.1.3).

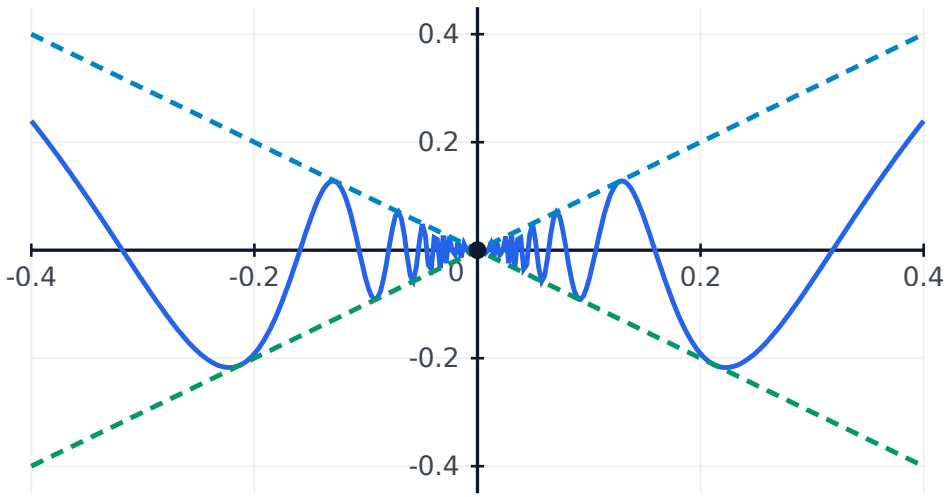


Figure 1.1.2: The squeeze theorem: $-|x| \leq x\sin(1/x) \leq |x|$, trapping the oscillating function at the origin.

For $x \neq 0$: $|f(x)| = |x|\,|\sin(1/x)| \leq |x| \cdot 1 = |x|$, since $|\sin t| \leq 1$ for all t. Both bound functions $\mp|x|$ tend to 0 as $x \to 0$, so the Squeeze Theorem (Theorem 1.1.3) gives $\lim_{x\to 0} f(x) = 0$. Since $f(0)$ is undefined but the limit exists, the extension $F(0) := 0$ makes F continuous at 0 (removable discontinuity). By contrast $\sin(1/x)$ has no limit at 0: its values cycle through $[-1, 1]$ in every neighborhood, so no candidate L survives $\epsilon = 1/2$.

Verification criteria. Requires: (a) the explicit bound $|\sin t| \leq 1$ invoked with the squeeze; (b) the conclusion stated for the *extended* function; (c) a falsifying ϵ (any $\epsilon < 1$ works) for $\sin(1/x)$. Claiming "$\sin(1/x)$ is undefined at 0, so no limit" is a **failure**: undefinedness at c does not preclude a limit.

Exercise 1.1.4: Guaranteed vs Unguaranteed Intermediate Values

Example / Impossible transfer

A continuous function g on $[-1, 1]$ satisfies $g(-1) = -0.5$ and $g(1) = 0.5$. For each value u, decide whether the Intermediate Value Theorem (Theorem 1.1.5) guarantees a solution of $g(x) = u$ on $[-1, 1]$: **(a)** $u = 0$; **(b)** $u = 0.9$; **(c)** $u = -0.5$.

Check each u against the hypothesis: is u between $g(-1)$ and $g(1)$?

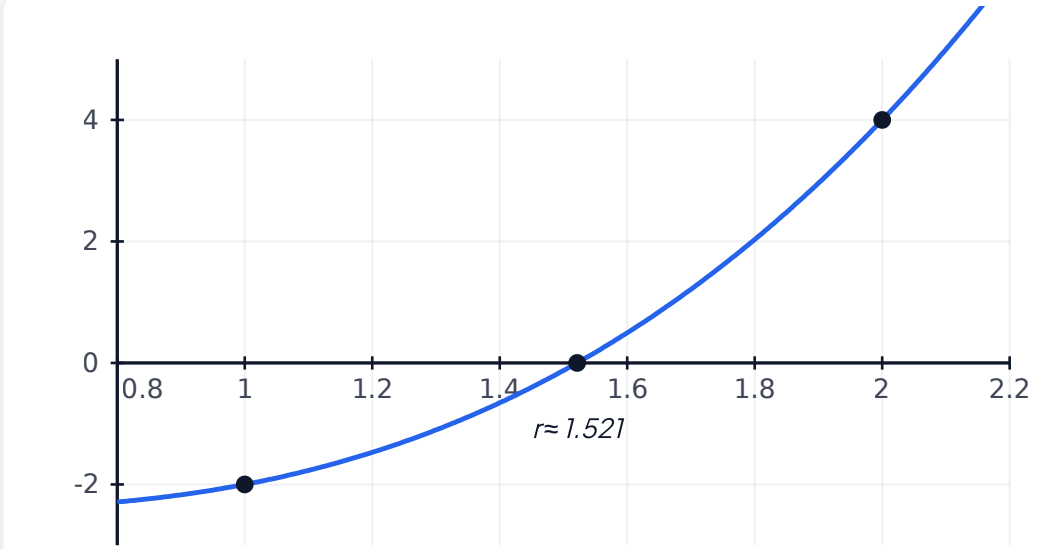


Figure 1.1.4: The Intermediate Value Theorem guarantees a root $r \approx 1.5214$ in $[1, 2]$ because $f(1) = -2 < 0 < 4 = f(2)$.

- **(a)** Yes: $-0.5 < 0 < 0.5$, so Theorem 1.1.5 guarantees some $c \in (-1, 1)$ with $g(c) = 0$.
- **(b)** No: 0.9 is not between $g(-1) = -0.5$ and $g(1) = 0.5$. The IVT is silent — a solution may or may not exist depending on g (if $g(x) = x$, none exists).
- **(c)** Yes — but for a different reason: $u = -0.5 = g(-1)$ is *attained at the endpoint*, not guaranteed inside (a, b). The IVT as stated locates interior points for values strictly between $f(a)$ and $f(b)$; endpoint values are already in the image by definition.

Verification criteria. Each part scores independently. For (a): correct yes/no with the hypothesis quoted. For (b): a bare "no" without the reason ("u not between the endpoint values") is incomplete; a "no, because g is discontinuous" is a **failure** (g is continuous by hypothesis). For (c): distinguishing *guaranteed by IVT* from *true but not via IVT* is the transfer step.

Exercise 1.1.5: Fixed-Point Existence via Intermediate Values

Proof synthesis

Prove, using Theorem 1.1.5, that the equation $x = \cos x$ has a solution in $[0, 1]$. Define the auxiliary function yourself; you may use that $\cos x$ is continuous on $[0, 1]$ with $\cos 0 = 1$ and $\cos 1 < 1$ ($\cos 1 \approx 0.54$).

Turn the fixed-point equation into a root-finding problem: move everything to one side and apply the IVT to the resulting continuous function.

Let $h(x) = x - \cos x$. As a difference of continuous functions (x and $\cos x$), h is continuous on $[0, 1]$. At the endpoints: $h(0) = 0 - 1 = -1 < 0$, and $h(1) = 1 - \cos 1 > 0$ since $\cos 1 < 1$. So $u = 0$ lies strictly between $h(0)$ and $h(1)$, and Theorem 1.1.5 guarantees some $c \in (0, 1)$ with $h(c) = 0$, i.e., $c = \cos c$.

Verification criteria. Requires: (a) an explicitly defined auxiliary function; (b) a stated continuity argument (sum/difference of continuous functions); (c) *strict* endpoint signs with justification ($\cos 1 < 1$, not "$\approx$"); (d) the IVT invoked on $u = 0$ between the endpoint values. Asserting "since $\cos$ is continuous it has a fixed point" without the sign change is a **failure** — continuity alone on a non-closed or badly-oriented domain does not give fixed points (compare $x \mapsto x + 1$).

Exercise 1.1.6: Continuity of Modified Dirichlet Function

Description transfer

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x$ if x is rational, and $f(x) = 0$ if x is irrational. At which points is f continuous according to Definition 1.1.4?

KNOWLEDGE CHECK

Let $f(x) = x$ for $x \in \mathbb{Q}$ and $f(x) = 0$ for $x \in \mathbb{Q}$. Evaluate each statement about continuity:

Answer: The function is continuous at $x = 0$ because $|f(x) - f(0)| \leq |x| < \epsilon$ whenever $|x| < \epsilon$.; The function is discontinuous at every nonzero point $c \neq 0$ due to conflicting rational and irrational sequences.

At $x = 0$: $|f(x) - f(0)| = |f(x)| \leq |x|$, since $f(x)$ is either x or 0; given $\epsilon > 0$, choose $\delta = \epsilon$, so f is continuous at 0. For any $c \neq 0$: approaching c through irrationals gives $f(x) = 0 \to 0$, while approaching through rationals gives $f(x) = x \to c \neq 0$; the two traces disagree, so no limit exists and f is discontinuous at c.

Exercise 1.1.7: Logical Negation of the Limit Definition

Negation recall

Formulate the precise logical negation of Definition 1.1.1 ($\lim_{x \to c} f(x) = L$) using quantifiers. Then apply your negated statement to prove that $\lim_{x \to 0} \operatorname{sgn}(x) \not= 0$.

Flipping '$\forall \epsilon > 0, \exists \delta > 0, \ldots$' yields '$\exists \epsilon_0 > 0, \forall \delta > 0, \ldots$'. Identify a specific ϵ_0 that $\operatorname{sgn}(x)$ always violates near 0.

Logical Negation: The statement $\lim_{x \to c} f(x) \not= L$ means: There exists $\epsilon_0 > 0$ such that for every $\delta > 0$, there exists at least one $x \in D$ satisfying:

$$0 < |x - c| < \delta \quad \text{and} \quad |f(x) - L| \geq \epsilon_0.$$

Application to $\operatorname{sgn}(x)$ at $c = 0$ with $L = 0$: Choose $\epsilon_0 = 1$. Let $\delta > 0$ be arbitrary. Select $x = \delta/2$. Then $0 < |x - 0| = \delta/2 < \delta$, and $x > 0$ so $\operatorname{sgn}(x) = 1$. We evaluate the distance:

$$|\operatorname{sgn}(x) - L| = |1 - 0| = 1 \geq \epsilon_0.$$

Since this holds for every $\delta > 0$, the condition holds and $\lim_{x \to 0} \operatorname{sgn}(x) \not= 0$.

Verification criteria. Requires: (a) correct quantifier inversion ($\forall \epsilon \to \exists \epsilon_0$, $\exists \delta \to \forall \delta$, and the conditional $P \implies Q$ negated to $P \wedge \neg Q$); (b) an explicit witness ϵ_0 (any $\epsilon_0 \leq 1$); (c) an explicit witness point x as a function of δ.

References & Further Reading

- Spivak, M. (2008). *Calculus* (4th ed.). Publish or Perish. — ϵ-δ proofs: Ch. 5 "Limits", §"Delta-Epsilon Proofs"; algebra of limits: Ch. 5, §"Theorems on Limits"; continuity: Ch. 6 "Continuous Functions"; boundedness/IVT/EVT: Ch. 7 "Three Hard Theorems", Thms 1–3; completeness proofs of IVT/EVT and uniform continuity (beyond this lesson's scope): Ch. 8 "Least Upper Bound".

- Rudin, W. (1976). *Principles of Mathematical Analysis* (3rd ed.). McGraw-Hill. — limits and continuity: Ch. 4, §"Limits of Functions" and §"Continuous Functions"; sequential criterion for limits (Thm 4.12), EVT (Thms 4.16–4.17), IVT (Thm 4.23), and metric-space/compactness generalizations.
- Burden, R. & Faires, J. (2011). *Numerical Analysis* (9th ed.). Cengage. §2.1 "The Bisection Method" — error bound and halving convergence.
- Abbott, S. (2015). *Understanding Analysis* (2nd ed.). Springer. — Ch. 4 "Functional Limits and Continuity": an alternate development of ϵ-δ limits (§4.2), the squeeze theorem, the IVT (§4.5), the EVT (§4.4), and uniform continuity on compact intervals.

SECTION DETAIL

Derivatives: Rates of Change

A particle's position along a line is modeled as $s(t) = \sqrt{t}$ metres after t seconds. For $h \neq 0$ with $4 + h \geq 0$, its average velocity between $t = 4$ and $t = 4 + h$ is

$$\frac{s(4+h) - s(4)}{h} = \frac{\sqrt{4+h} - 2}{h}.$$

At $h = 0$ this expression becomes $0/0$, so substituting zero does not give an instantaneous velocity. Instead, rationalizing for $h \neq 0$ gives $1/(\sqrt{4+h} + 2)$. This approaches $1/4$ as h approaches zero from either side: the limiting rate is $1/4$ m/s. The derivative formalizes this limiting process and, geometrically, gives the slope of a tangent line.

Differentiability and the Derivative

We formally define the derivative of a function at a point as a limit (Definition 1.1.1) of the difference quotient.

DEFINITION 1.2.1 **The Derivative**

Let $f : I \to \mathbb{R}$ be defined on an open interval I containing c. The function f is said to be **differentiable** at c if the limit

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists. If this limit exists, we denote it by $f'(c)$ or $\frac{df}{dx}(c)$.

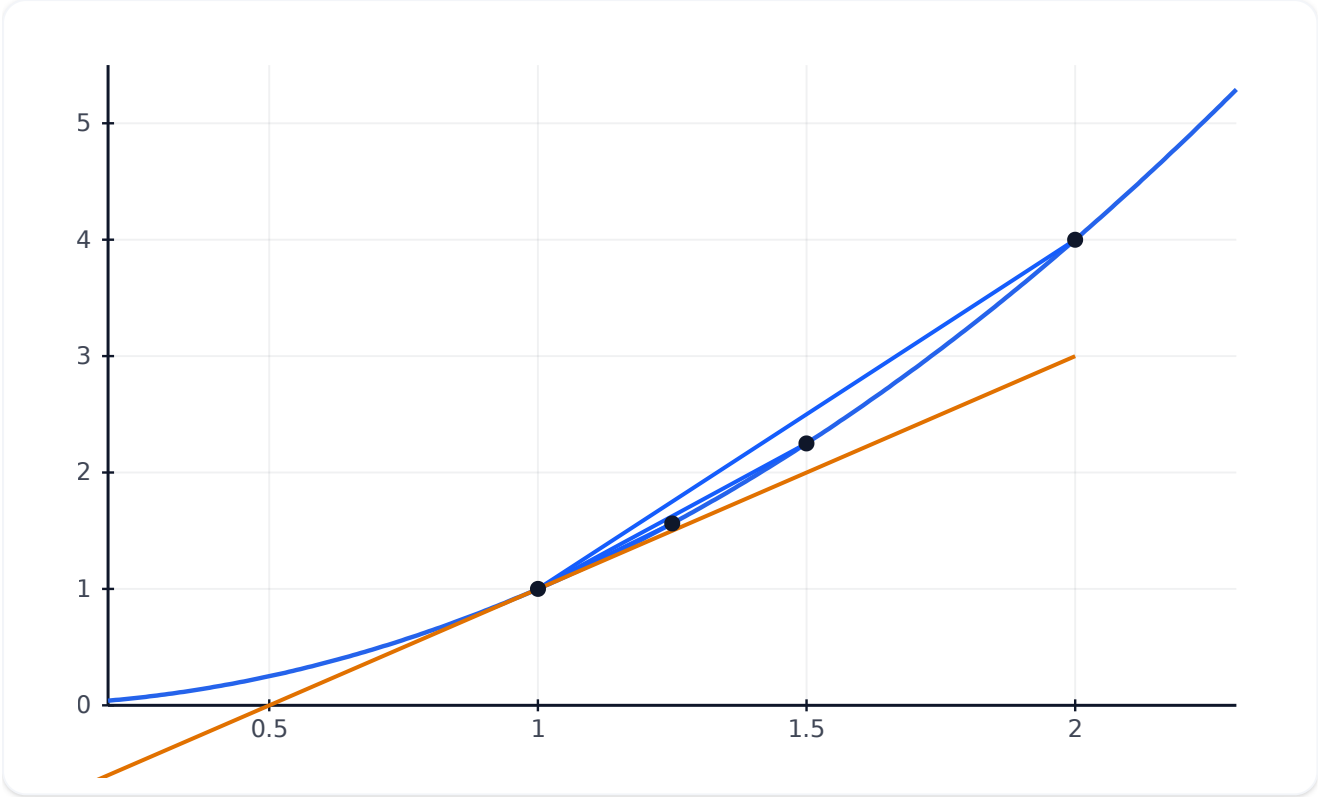


Figure 1.2.1: Secant lines of $y = x^2$ at $a = 1$ converge to the tangent line of slope $f'(1) = 2$ as $h \to 0$.

Differentiability is a stronger condition than continuity (Definition 1.1.4). A function can be continuous but not differentiable, but the converse is always true.

THEOREM 1.2.2 **Differentiability Implies Continuity**

Builds on: Definition 1.2.1

If f is differentiable at c, then f is continuous at c.

To prove $\lim_{x \to c} f(x) = f(c)$, it suffices to show that $\lim_{x \to c}(f(x) - f(c)) = 0$. For $x \neq c$, we can write:

$$f(x) - f(c) = \frac{f(x) - f(c)}{x - c} \cdot (x - c).$$

Taking the limit as $x \to c$:

$$\lim_{x \to c}(f(x) - f(c)) = \lim_{x \to c}\left(\frac{f(x) - f(c)}{x - c} \cdot (x - c)\right).$$

Since f is differentiable at c, the limit of the first term is $f'(c)$. Since $\lim_{x \to c}(x - c) = 0$, we have:

$$\lim_{x \to c}(f(x) - f(c)) = f'(c) \cdot 0 = 0.$$

Therefore, $\lim_{x \to c} f(x) = f(c)$, and f is continuous at c.

A classic counterexample of a function that is continuous but not differentiable is $f(x) = |x|$ at $x = 0$. The difference quotient as $h \to 0^+$ is 1, but as $h \to 0^-$ is -1, so the limit does not exist.

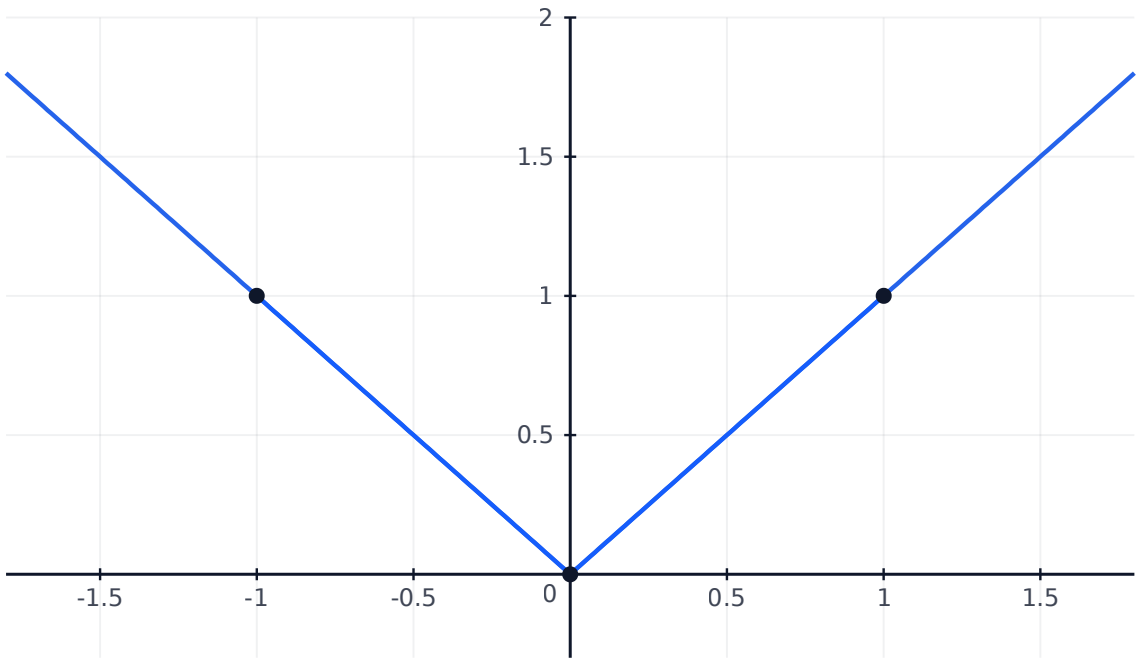


Figure 1.2.2: Non-differentiability at a corner: left secant slope is -1 while right secant slope is $+1$, so no tangent exists at $x = 0$.

Worked trace: derivative from the definition via rationalization. Let $f(x) = \sqrt{x}$. Find $f'(4)$ using the limit definition.

1. Form difference quotient: $\frac{\sqrt{4+h}-\sqrt{4}}{h} = \frac{\sqrt{4+h}-2}{h}$.
2. Multiply numerator and denominator by conjugate $(\sqrt{4+h}+2)$:

$$\frac{(\sqrt{4+h}-2)(\sqrt{4+h}+2)}{h(\sqrt{4+h}+2)} = \frac{(4+h)-4}{h(\sqrt{4+h}+2)} = \frac{h}{h(\sqrt{4+h}+2)}.$$

3. Cancel $h \neq 0$:

$$\frac{1}{\sqrt{4+h}+2}.$$

4. Evaluate limit as $h \to 0$:

$$f'(4) = \lim_{h\to 0} \frac{1}{\sqrt{4+h}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}.$$

The calculation confirms the particle's instantaneous velocity at $t = 4$: $1/4$ m/s. Cancel the factor h only for $h \neq 0$ before taking the limit; the quotient is not defined at $h = 0$.

Worked trace: a reciprocal function and a domain boundary. Let $g(x) = 1/x$ for $x \neq 0$. At $x = 2$,

$$g'(2) = \lim_{h\to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h} = \lim_{h\to 0} \frac{\frac{2-(2+h)}{2(2+h)}}{h} = \lim_{h\to 0} \frac{-h}{2h(2+h)} = \lim_{h\to 0} \frac{-1}{2(2+h)} = -\frac{1}{4}.$$

Two features of this trace generalize. First, the same cancellation pattern appears whenever the numerator has a factor h; the algebra exposes the removable discontinuity rather than hiding it. Second, the computation is valid only at points where g is defined on a two-sided neighbourhood. At $x = 0$ the function is undefined, and no difference quotient exists, so "$g'(0)$" is not a number to be found—it is a question with no domain.

Worked trace: a continuous function that fails differentiability. Let

$$f(x) = \begin{cases} x\sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Continuity at 0 follows from $|x\sin(1/x)| \le |x|$. The difference quotient is

$$\frac{f(h)-f(0)}{h} = \sin(1/h).$$

As $h \to 0$, $\sin(1/h)$ oscillates between -1 and 1 and has no limit. The function is continuous at 0 but not differentiable there. The counterexample shows that the converse of Theorem 1.2.2 is false and that a derivative can fail to exist even when the graph has no jump or corner in the usual pictorial sense.

KNOWLEDGE CHECK

Consider $f(x) = x^2\sin(1/x)$ for $x \neq 0$, with $f(0) = 0$. Evaluate each statement regarding differentiability and derivative continuity at $x = 0$:

Answer: The function is differentiable at $x = 0$ with derivative value $f'(0) = 0$ by the squeeze theorem.; The derivative $f'(x)$ exists everywhere but fails to be continuous at $x = 0$ due to oscillation.

Using the limit definition, $f'(0) = \lim_{h\to 0} \frac{h^2\sin(1/h)}{h} = \lim_{h\to 0} h\sin(1/h) = 0$ by $|h\sin(1/h)| \le |h|$. For $x \neq 0$, $f'(x) = 2x\sin(1/x) - \cos(1/x)$, which has no limit at 0; differentiability does not imply continuity of the derivative.

Fundamental Rules of Differentiation

> ***Pause & Predict: Tangent Slopes Along a Cubic Curve.*** *In the interactive playground below, you can move the point of tangency a along the curve $f(x) = 0.16x^3 - 0.5x$. At what values of a does the tangent line become horizontal ($f'(a) = 0$)? Where is the slope positive ($f'(a) > 0$), and where is it negative ($f'(a) < 0$)? Predict the coordinates of the horizontal tangents before adjusting the slider.*

Differentiating $f(x) = 0.16x^3 - 0.5x$ using the power and sum rules:

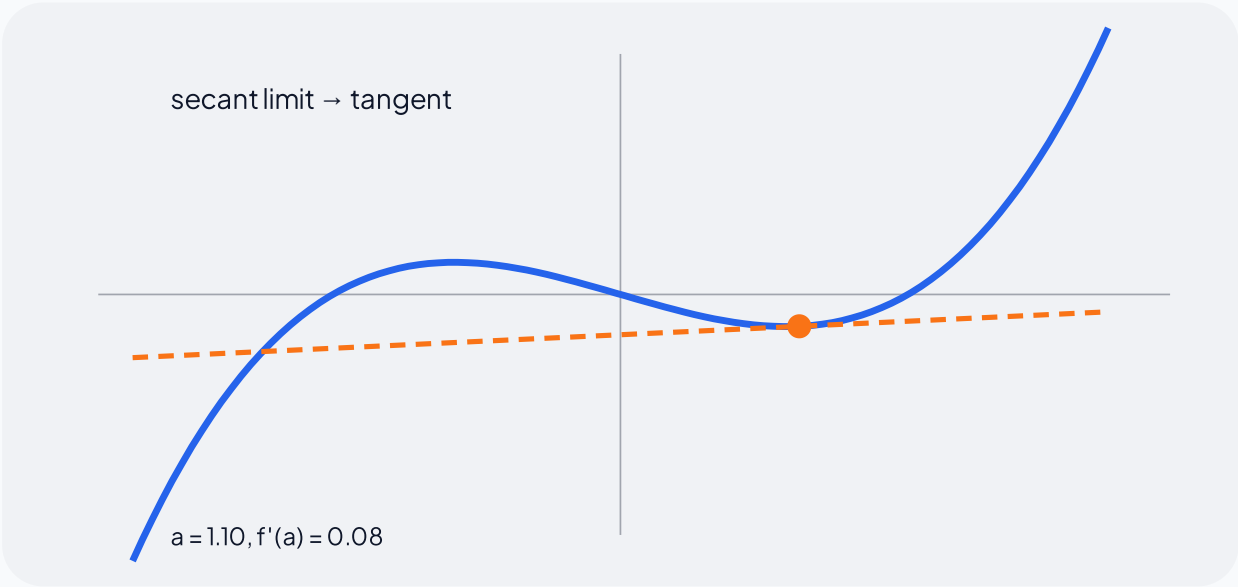
$$f'(x) = 3(0.16)x^2 - 0.5 = 0.48x^2 - 0.5.$$

Setting $f'(a) = 0$:

$$0.48a^2 - 0.5 = 0 \implies a^2 = \frac{0.5}{0.48} \approx 1.0417 \implies a \approx \pm 1.02.$$

- For $|a| > 1.02$, $f'(a) > 0$ (the curve is strictly increasing).
- For $|a| < 1.02$, $f'(a) < 0$ (the curve is strictly decreasing).
- At $a \approx -1.02$ (local maximum) and $a \approx 1.02$ (local minimum), the tangent line is horizontal.

SLIDE THE POINT OF TANGENCY



A tangent line records the limiting slope at a selected point.

To compute derivatives efficiently, we establish algebraic rules derived from the definition.

THEOREM 1.2.3 **Differentiation Rules**

Builds on: Definition 1.2.1

Let f and g be differentiable at x.

1. **Sum Rule**: $(f+g)'(x) = f'(x) + g'(x)$.
2. **Product Rule**: $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.
3. **Quotient Rule**: If $g(x) \neq 0$, $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$.

We prove the Product Rule. Let $F(x) = f(x)g(x)$. Using the definition:

$$F'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

Subtract and add the term $f(x)g(x+h)$ in the numerator:

$$F'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \cdot g(x+h) + f(x) \cdot \frac{g(x+h) - g(x)}{h} \right).$$

Since f and g are differentiable, their individual limits exist. Since g is differentiable, it is continuous, so $\lim_{h \to 0} g(x+h) = g(x)$. Applying limit laws:

$$F'(x) = f'(x)g(x) + f(x)g'(x).$$

∎

The Chain Rule

The Chain Rule allows us to differentiate composite functions. It is the mathematical engine behind backpropagation in deep neural networks.

THEOREM 1.2.4 **The Chain Rule**

Builds on: Definition 1.2.1

If g is differentiable at x, and f is differentiable at $g(x)$, then the composite function $y = (f \circ g)(x) = f(g(x))$ is differentiable at x, and:

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x).$$

Linear Approximation

The derivative provides the best linear approximation of a function near a point. Geometrically, this is the tangent line equation.

$$f(x) \approx f(a) + f'(a)(x - a).$$

REMARK

Single-variable differentiation treats the derivative as a scalar slope $f'(c) \in \mathbb{R}$; in higher dimensions, the derivative is the unique linear map $Df(c) : \mathbb{R}^n \to \mathbb{R}^m$ satisfying $\lim_{h \to 0} \frac{\|f(c+h) - f(c) - Df(c)h\|}{\|h\|} = 0$.

For example, linearising $f(x) = \sqrt{x}$ at $a = 4$ gives $L(x) = 2 + \frac{1}{4}(x - 4)$, which closely tracks $\sqrt{x}$ near $x = 4$ while gradually diverging further away:

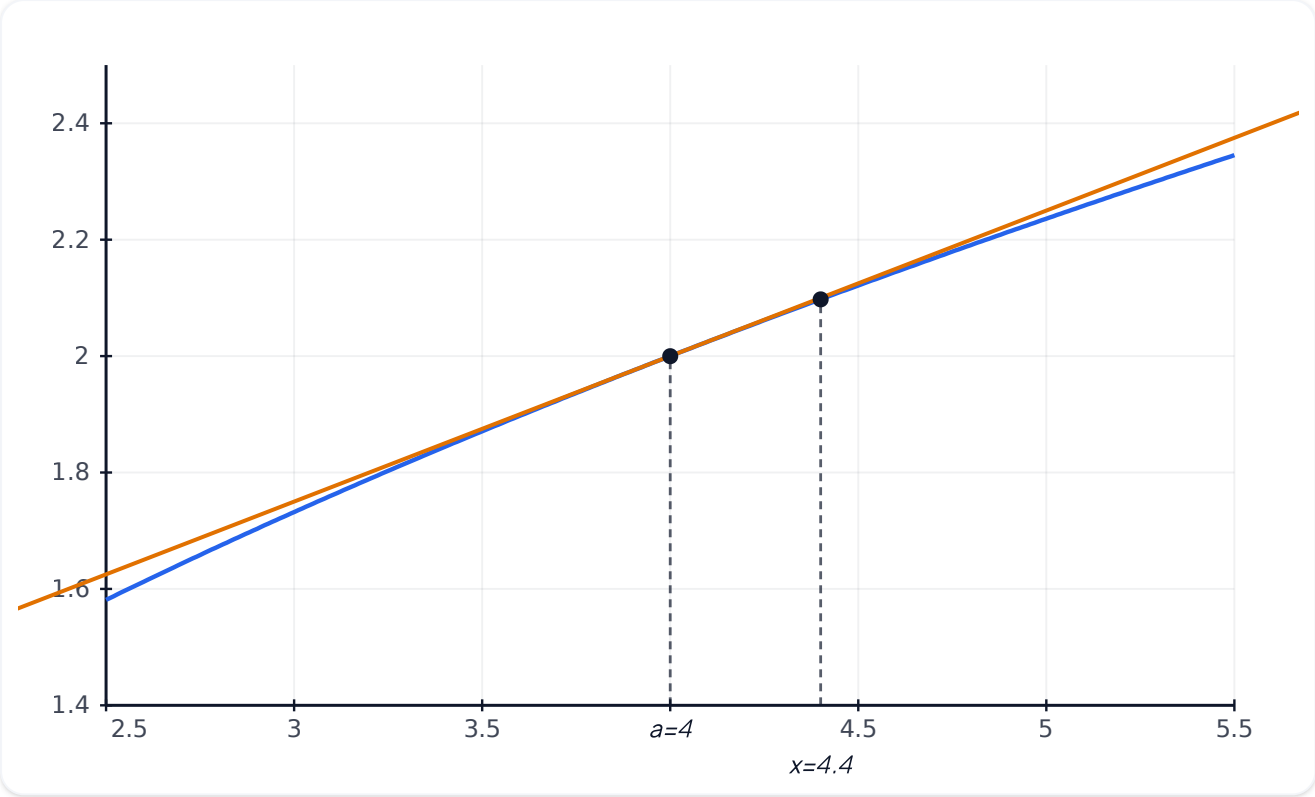


Figure 1.2.3: Linear approximation $L(x) = 2 + \frac{x-4}{4}$ to $\sqrt{x}$ at $a = 4$: at $x = 4.4, L(4.4) = 2.100$ exceeds $\sqrt{4.4} \approx 2.098$ by 0.0024.

Let us visualize how the linear approximation matches $f(x) = \sin(x)$ around $a = 1.0$, and verify the error increases as we move away from a.

INTERACTIVE LAB

```
import numpy as np
import matplotlib.pyplot as plt

def f(x): return np.sin(x)
def df(x): return np.cos(x)

a = 1.0
x = np.linspace(0.0, 2.0, 200)
l_approx = f(a) + df(a) * (x - a)

plt.plot(x, f(x), label='f(x) = sin(x)', color='#3b82f6', linewidth=2)
plt.plot(x, l_approx, '--', label='Linear Approx at x=1', color='r')
plt.scatter([a], [f(a)], color='black', zorder=5, label='Point of Tangency')
plt.legend()
plt.title("Tangent Line (Linear Approximation)")
plt.grid(True)
plt.show()

# Compute error at x = 1.1 and x = 1.8
print(f"Error at x = 1.1: {abs(f(1.1) - (f(a) + df(a)*(1.1-a))):.6f}")
print(f"Error at x = 1.8: {abs(f(1.8) - (f(a) + df(a)*(1.8-a))):.6f}")
```

Implicit Differentiation

When variables are defined implicitly (e.g., $x^2 + y^2 = 1$), we differentiate both sides with respect to x, applying the Chain Rule to y (treating y locally as a differentiable function of x).

For the unit circle $x^2 + y^2 = 1$, differentiating yields:

$$2x + 2y \cdot \frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{x}{y} \quad (y \neq 0).$$

EXAMPLE

On the circle $x^2 + y^2 = 25$ at $(3, 4)$, the slope is $\frac{dy}{dx}\big|_{(3,4)} = -\frac{3}{4}$, giving the tangent line $y - 4 = -\frac{3}{4}(x - 3)$, or $3x + 4y = 25$. At the equator points $(\pm 5, 0)$, the denominator $y = 0$ vanishes: the tangent line is vertical ($x = \pm 5$), and y cannot be written as a differentiable function of x on any neighborhood containing $(\pm 5, 0)$.

1. Differentiate both sides with respect to x, using the Product Rule on $6xy$:

$$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}.$$

2. Collect $\frac{dy}{dx}$ terms:

$$(3y^2 - 6x)\frac{dy}{dx} = 6y - 3x^2 \implies \frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}.$$

3. Evaluate at $(3, 3)$:

$$\frac{dy}{dx}\Big|_{(3,3)} = \frac{2(3) - 3^2}{3^2 - 2(3)} = \frac{6 - 9}{9 - 6} = -1.$$

SECTION EXERCISES

Deep Practice

Apply differentiation rules, evaluate differentiability contracts, and explore the boundary between derivative existence and derivative continuity.

Exercise 1.2.1: Derivative from First Principles

Verification application

Using Definition 1.2.1, compute the derivative of $f(x) = \frac{1}{\sqrt{x}}$ at an arbitrary point $x > 0$.

Form the difference quotient $\frac{1/\sqrt{x+h} - 1/\sqrt{x}}{h}$, find a common denominator, and rationalize the numerator.

1. Form the difference quotient:

$$\frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}.$$

2. Multiply numerator and denominator by the conjugate $(\sqrt{x} + \sqrt{x+h})$:

$$\frac{(\sqrt{x} - \sqrt{x+h})(\sqrt{x} + \sqrt{x+h})}{h\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})} = \frac{x - (x+h)}{h\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})} = \frac{}{h\sqrt{x}\sqrt{x}}$$

3. Cancel $h \neq 0$:

$$\frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})}.$$

4. Take the limit as $h \to 0$ using limit laws:

$$f'(x) = \frac{-1}{\sqrt{x}\sqrt{x}(\sqrt{x} + \sqrt{x})} = \frac{-1}{x(2\sqrt{x})} = -\frac{1}{2x^{3/2}}.$$

Verification criteria. Requires: (a) explicit difference quotient; (b) algebraic conjugate rationalization; (c) valid cancellation of h for $h \neq 0$; (d) correct final algebraic term $-1/(2x^{3/2})$.

Exercise 1.2.2: Continuity Without Differentiability

Negation recall

Show that $f(x) = |x|$ is continuous at $x = 0$ by Theorem 1.2.2's converse, but not differentiable at $x = 0$ by Definition 1.2.1.

Evaluate the left-hand and right-hand limits of the difference quotient $(f(0+h)-f(0))/h$.

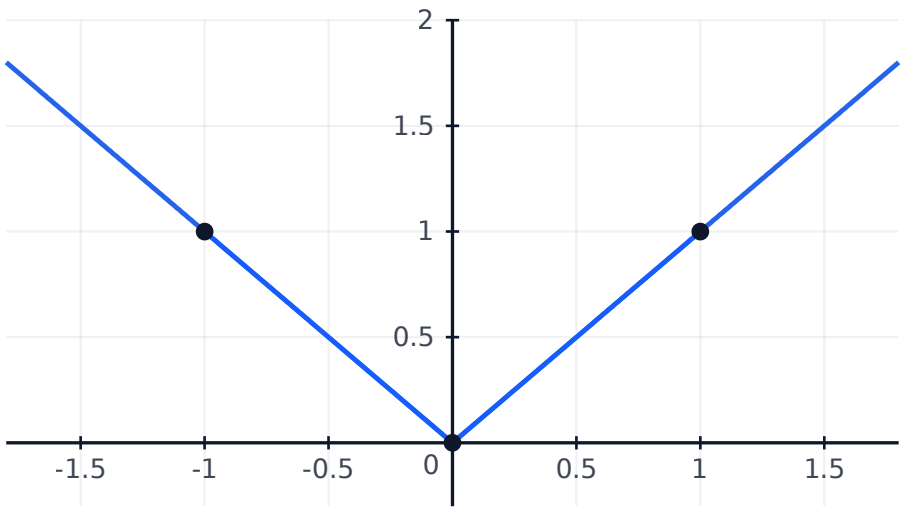


Figure 1.2.2: Non-differentiability at a corner: left secant slope is -1 while right secant slope is $+1$, so no tangent exists at $x = 0$.

Continuity: $f(0) = 0$. Since $|f(x) - f(0)| = ||x| - 0| = |x| < \epsilon$ whenever $|x - 0| < \delta = \epsilon$, $\lim_{x \to 0} f(x) = f(0) = 0$. Thus f is continuous at 0.

Non-Differentiability: The difference quotient at $c = 0$ is:

$$\frac{f(0+h) - f(0)}{h} = \frac{|h| - 0}{h} = \frac{|h|}{h}.$$

As $h \to 0^+$, $|h| = h$, so $\lim_{h \to 0^+} \frac{h}{h} = 1$. As $h \to 0^-$, $|h| = -h$, so $\lim_{h \to 0^-} \frac{-h}{h} = -1$. Since the one-sided limits disagree ($1 \neq -1$), the two-sided limit does not exist. Hence f is not differentiable at 0. This confirms the converse of Theorem 1.2.2 is false.

Verification criteria. Requires computing both one-sided difference quotient limits and concluding divergence.

Exercise 1.2.3: Product and Quotient Rules on Algebraic Forms

Construction application

Using Theorem 1.2.3, find $f'(x)$ for $f(x) = \frac{x^2 \sin x}{x^2+1}$.

Apply the Quotient Rule with numerator $u(x) = x^2 \sin x$ (which requires the Product Rule) and denominator $v(x) = x^2 + 1$.

1. Let $u(x) = x^2 \sin x$. By the Product Rule:

$$u'(x) = (x^2)' \sin x + x^2 (\sin x)' = 2x \sin x + x^2 \cos x.$$

2. Let $v(x) = x^2 + 1$. Its derivative is $v'(x) = 2x$.
3. Applying the Quotient Rule:

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2} = \frac{(2x \sin x + x^2 \cos x)(x^2 + 1) - (x^2 \sin x}{(x^2 + 1)^2}$$

4. Expanding the numerator:

$$(2x^3 \sin x + 2x \sin x + x^4 \cos x + x^2 \cos x) - 2x^3 \sin x = x^2(x^2 + 1) \cos x$$

Thus:

$$f'(x) = \frac{x(x(x^2 + 1) \cos x + 2 \sin x)}{(x^2 + 1)^2}.$$

Verification criteria. A complete solution requires correct composition of product and quotient rules and simplification.

Exercise 1.2.4: Linear Approximation Error Bound

Construction application

Let $f(x) = \sqrt{x}$. Find the linear approximation $L(x)$ of f at $a = 25$. Use $L(x)$ to estimate $\sqrt{26}$, and show without a calculator that the error is less than $1/500$.

Recall $L(x) = f(a) + f'(a)(x - a)$. For $f(x) = \sqrt{x}$, $f'(25) = 1/(2\sqrt{25}) = 1/10$.

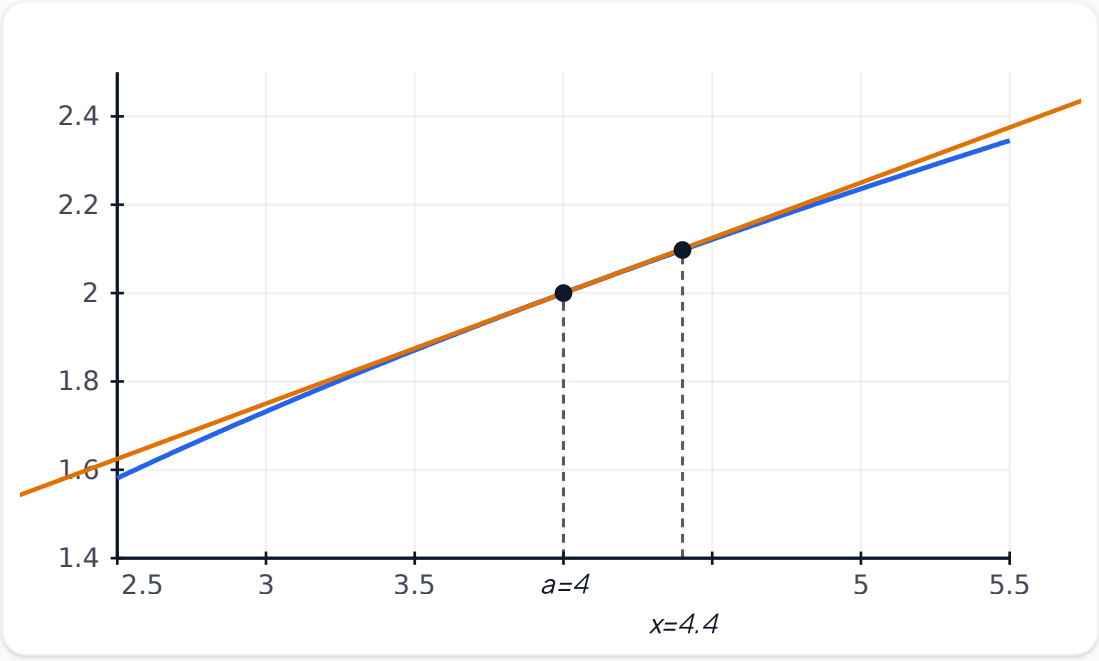


Figure 1.2.3: Linear approximation $L(x) = 2 + \frac{x-4}{4}$ to $\sqrt{x}$ at $a = 4$: at $x = 4.4$, $L(4.4) = 2.100$ exceeds $\sqrt{4.4} \approx 2.098$ by 0.0024.

1. At $a = 25$: $f(25) = 5$, and $f'(x) = \frac{1}{2\sqrt{x}}$, so $f'(25) = \frac{1}{10} = 0.1$.
2. The linear approximation is:

$$L(x) = 5 + \frac{1}{10}(x - 25).$$

3. At $x = 26$:

$$L(26) = 5 + \frac{1}{10}(26 - 25) = 5 + 0.1 = 5.1.$$

4. **Error analysis:** Squaring 5.1 gives $(5.1)^2 = 26.01$. Since $26 < 26.01$, $\sqrt{26} < 5.1$. Also $(5.099)^2 = (5.1 - 0.001)^2 = 26.01 - 0.0102 + 0.000001 = 25.999801 < 26$. Thus $5.099 < \sqrt{26} < 5.100$, confirming $|\sqrt{26} - 5.1| < 0.001 < \frac{1}{500} = 0.002$.

Verification criteria. Requires the explicit tangent line equation, the estimate 5.1, and the rigorous square comparison verifying the error bound.

Exercise 1.2.5: Implicit Differentiation and Normal Slopes

Construction application

Consider the Folium of Descartes $x^3 + y^3 = 6xy$. Find the equations of all horizontal and vertical tangent lines on the curve where $x \neq 0$ and $y \neq 0$.

From the worked trace, $\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$. Horizontal tangents have numerator $= 0$; vertical tangents have denominator $= 0$.

1. **Horizontal Tangents:** $\frac{dy}{dx} = 0 \implies 2y - x^2 = 0 \implies y = \frac{x^2}{2}$. Substitute into $x^3 + y^3 = 6xy$:

$$x^3 + \left(\frac{x^2}{2}\right)^3 = 6x\left(\frac{x^2}{2}\right) \implies x^3 + \frac{x^6}{8} = 3x^3 \implies \frac{x^6}{8} = 2x^3.$$

   Since $x \neq 0$, divide by x^3:

$$x^3 = 16 \implies x = 2^{4/3} = 2\sqrt[3]{2}.$$

   Then $y = \frac{x^2}{2} = \frac{2^{8/3}}{2} = 2^{5/3} = 2\sqrt[3]{4} = 4^{2/3}$. The horizontal tangent line is $y = 2\sqrt[3]{4}$.

2. **Vertical Tangents:** By symmetry (swapping x and y leaves $x^3 + y^3 = 6xy$ unchanged), the denominator vanishes when $y^2 - 2x = 0$, giving the vertical tangent at $x = 2\sqrt[3]{4}$.

Verification criteria. Requires setting numerator/denominator to zero, solving with the curve equation, and identifying the coordinates.

Exercise 1.2.6: Derivation of the Inverse Function Rule

Generalisation synthesis

Let f be strictly monotonic and differentiable on an interval I, with $f'(x) \neq 0$. Let $g = f^{-1}$ be its inverse. Use Theorem 1.2.4 on the identity $f(g(y)) = y$ to prove that $g'(y) = \frac{1}{f'(g(y))}$.

Differentiate both sides of $f(g(y)) = y$ with respect to y, assuming g is differentiable.

1. The inverse function identity states:

$$f(g(y)) = y \quad \text{for all } y \text{ in the range of } f.$$

2. Differentiate both sides with respect to y, applying the Chain Rule (Theorem 1.2.4) to the composite left-hand side:

$$\frac{d}{dy}[f(g(y))] = \frac{d}{dy}[y].$$

3. By Theorem 1.2.4:

$$f'(g(y)) \cdot g'(y) = 1.$$

4. Since $f'(x) \neq 0$ for all x, $f'(g(y)) \neq 0$. Dividing gives:

$$g'(y) = \frac{1}{f'(g(y))}.$$

   Equivalently, setting $y = f(x)$ so $g(y) = x$:

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}.$$

Verification criteria. A complete derivation requires correct application of chain rule to identity and explicitly noting $f'(g(y)) \neq 0$.

References & Further Reading

- Michael Spivak (2008). *Calculus* (4th ed.), Chapter 9 ("Derivatives", pp. 143–162) and Chapter 10 ("Differentiation", pp. 165–181); for the construction of a continuous nowhere-differentiable function, see Chapter 24. Publish or Perish.
- Tom M. Apostol (1967). *Calculus, Volume 1: One-Variable Calculus with an Introduction to Linear Algebra* (2nd ed.), Chapter 4 ("Differential Calculus", pp. 155–182). John Wiley & Sons.

- Walter Rudin (1976). *Principles of Mathematical Analysis* (3rd ed.), Chapter 5 ("Differentiation", pp. 103–118), and Chapter 9 ("Functions of Several Variables") for the total derivative $Df(c)$ on $\mathbb{R}^n$. McGraw-Hill.

SECTION DETAIL

Mean Value Theorem: The Guarantee of Speed

The Mean Value Theorem (MVT) is one of the most critical theoretical tools in calculus. It establishes a direct connection between the average rate of change over an interval (global behavior) and the instantaneous rate of change at a point within that interval (local behavior).

Rolle's Theorem: The Foundation

Before proving the general Mean Value Theorem, we must prove its special case, Rolle's Theorem.

THEOREM 1.3.1 **Rolle's Theorem**

Builds on: Theorem 1.1.6

Let $f : [a, b] \to \mathbb{R}$ be a function that satisfies:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$.

Then there exists at least one $c \in (a, b)$ such that:

$$f'(c) = 0.$$

By the Extreme Value Theorem (Theorem 1.1.6), the continuous function f on the compact interval $[a, b]$ attains a global maximum M and a global minimum m:

$$\exists\, x_{\min}, x_{\max} \in [a,b] \quad \text{such that} \quad m = f(x_{\min}) \leq f(x) \leq f(x_{\max}) = M \quad \forall x \in [a,$$

Case 1: Constant function ($M = m$).

If $M = m$, then f is constant throughout $[a, b]$. Therefore:

$$f'(x) = 0 \quad \text{for all } x \in (a,b),$$

and *every* interior point $c \in (a, b)$ satisfies the theorem.

Case 2: Non-trivial extremum in the interior ($M \neq m$).

Because $f(a) = f(b)$, at least one of the extrema must occur at an interior point $c \in (a, b)$. Without loss of generality, suppose the maximum occurs in the interior: $f(c) = M$ for some $c \in (a, b)$.

Since $f(c) \geq f(x)$ for all $x \in [a, b]$, the numerator $f(x) - f(c) \leq 0$ everywhere. We evaluate the one-sided difference quotients:

1. **Right-hand derivative** ($x > c$): Here $x - c > 0$, so dividing non-positive by positive yields a non-positive ratio:

$$\frac{f(x) - f(c)}{x - c} \leq 0 \implies f'_+(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0.$$

2. **Left-hand derivative** ($x < c$): Here $x - c < 0$, so dividing non-positive by negative reverses the inequality:

$$\frac{f(x) - f(c)}{x - c} \geq 0 \implies f'_-(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0.$$

Since f is differentiable at c, both one-sided limits must agree with the two-sided derivative:

$$f'(c) = f'_+(c) = f'_-(c).$$

The only real value that satisfies both $f'(c) \leq 0$ and $f'(c) \geq 0$ is:

$$f'(c) = 0.$$

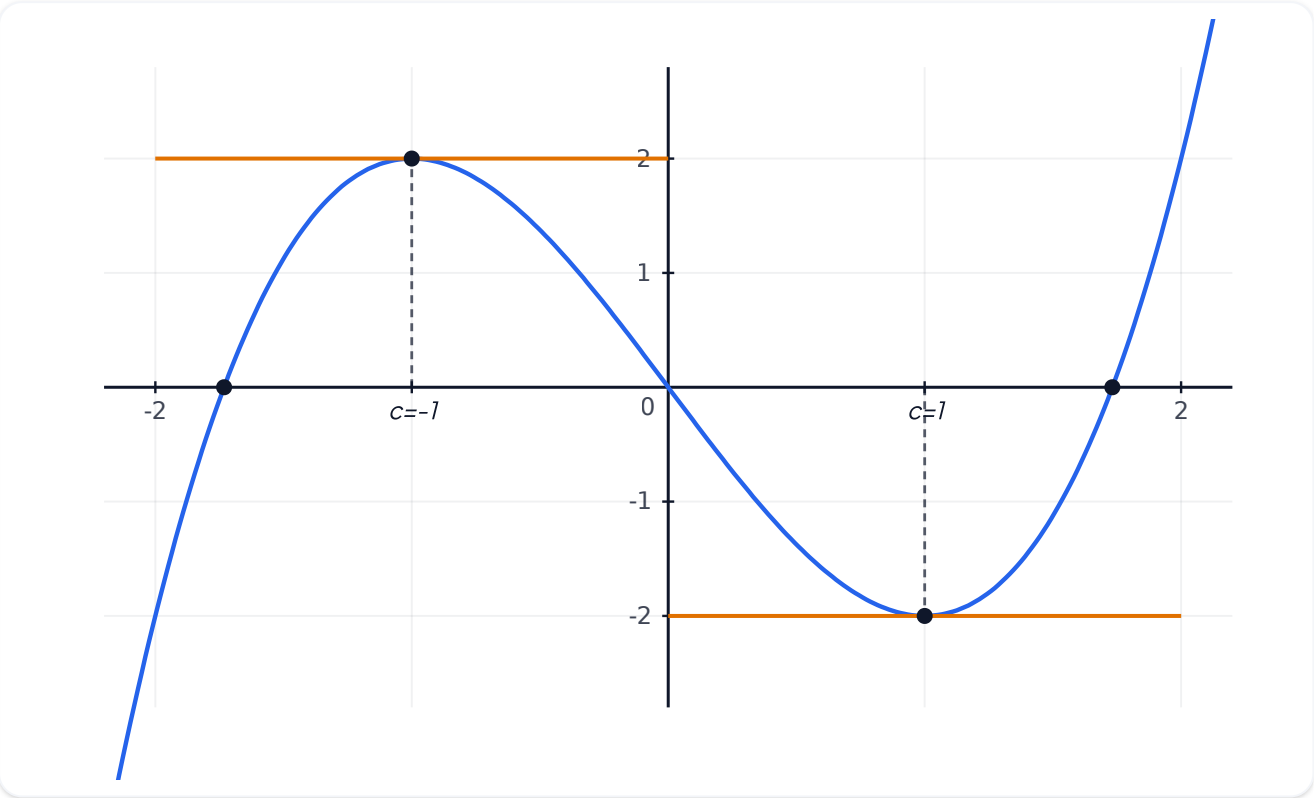


Figure 1.3.1: Rolle's theorem: $f(-\sqrt{3}) = f(\sqrt{3}) = 0$ implies $f'(c) = 0$ at both critical points $c = -1$ and $c = 1$.

The Mean Value Theorem

We generalize Rolle's Theorem to functions where $f(a) \neq f(b)$ by tilting the coordinate system.

THEOREM 1.3.2 **Mean Value Theorem (MVT)**

Builds on: Theorem 1.3.1

Let $f : [a, b] \to \mathbb{R}$ be a function satisfying:

1. f is continuous on $[a, b]$.
2. f is differentiable on (a, b).

Then there exists at least one $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

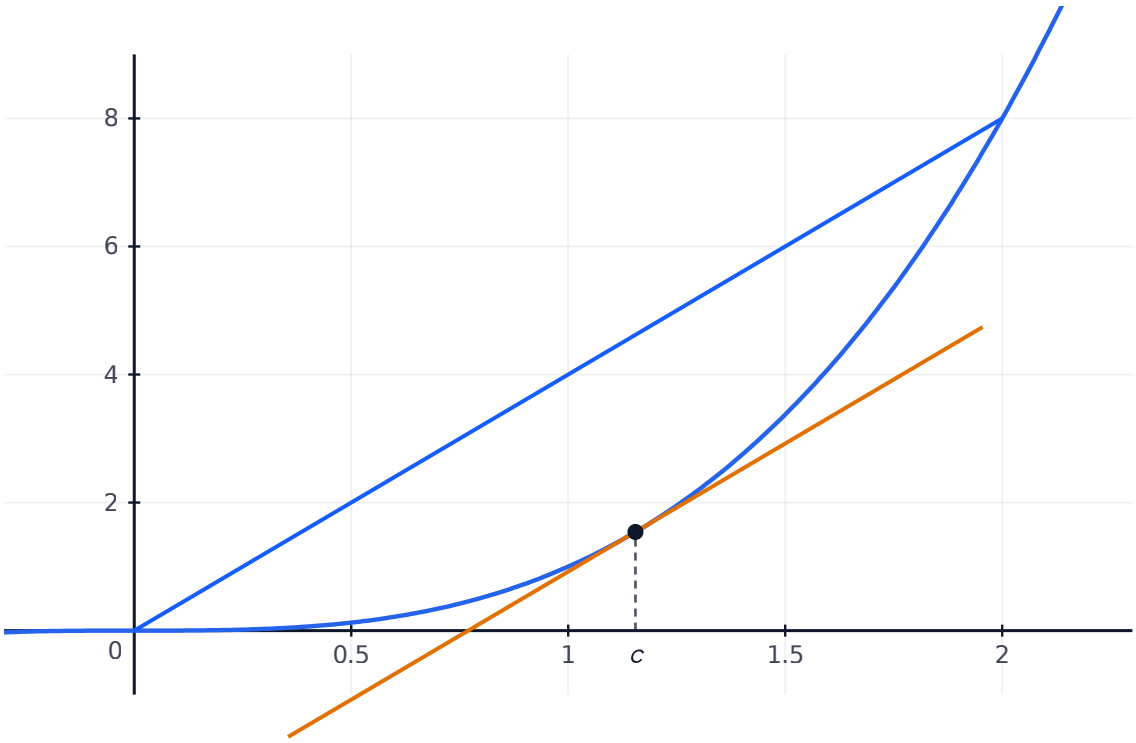


Figure 1.3.2: The tangent at c is parallel to the chord: $f'(c) = \frac{f(2) - f(0)}{2} = 4$.

Pause & Predict: Does the MVT extend to Vector-Valued Functions? *Suppose* $\mathbf{f} : [a, b] \to \mathbb{R}^2$ *is smooth and continuous. Does there necessarily exist* $c \in (a, b)$ *such that* $\mathbf{f}'(c) = \frac{\mathbf{f}(b) - \mathbf{f}(a)}{b - a}$*? Think of a particle moving along a circular path before opening the analysis below.*

Consider a particle travelling in a full circle $\mathbf{f}(t) = (\cos t, \sin t)$ on $[0, 2\pi]$. What is the average velocity vector, and does the particle ever come to a complete stop?

Analysis: No! The Mean Value Theorem fails for vector-valued functions $\mathbf{f} : \mathbb{R} \to \mathbb{R}^n$ $(n \geq 2)$. Consider $\mathbf{f}(t) = (\cos t, \sin t)$ on $[0, 2\pi]$. The average velocity over one full orbit is:

$$\frac{\mathbf{f}(2\pi) - \mathbf{f}(0)}{2\pi - 0} = \frac{(1,0) - (1,0)}{2\pi} = (0,0).$$

However, the velocity vector is $\mathbf{f}'(t) = (-\sin t, \cos t)$, whose magnitude is $\|\mathbf{f}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$ for all t. The instantaneous velocity never vanishes! In higher dimensions, one can only obtain the *Mean Value Inequality*: $\|\mathbf{f}(b) - \mathbf{f}(a)\| \leq (b-a) \sup_{t \in (a,b)} \|\mathbf{f}'(t)\|$.

We construct an auxiliary function $g(x)$ measuring the vertical difference between the curve $y = f(x)$ and the secant chord passing through $(a, f(a))$ and $(b, f(b))$:

$$g(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right].$$

We check the three hypotheses of Rolle's Theorem (Theorem 1.3.1) on $[a, b]$:

1. **Continuity:** g is continuous on $[a, b]$ because f is continuous and the linear chord is continuous.
2. **Differentiability:** g is differentiable on (a, b) because f is differentiable and linear terms are smooth.
3. **Boundary conditions:** Evaluating g at both endpoints:

$$g(a) = f(a) - [f(a) + 0] = 0,$$

$$g(b) = f(b) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(b - a) \right] = f(b) - f(b) = 0.$$

Since $g(a) = g(b) = 0$, Rolle's Theorem guarantees that there exists at least one $c \in (a, b)$ such that:

$$g'(c) = 0.$$

Differentiating $g(x)$ directly:

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

Evaluating at $x = c$:

$$0 = f'(c) - \frac{f(b) - f(a)}{b - a} \implies f'(c) = \frac{f(b) - f(a)}{b - a}.$$

■

The theorem identifies a point where the tangent slope equals the secant slope; the interactive Python visualization below lets us inspect that relationship directly. Let us write a Python script to verify the MVT for $f(x) = x^3 - 3x$ on $[0, 2]$. The secant slope is $(f(2) - f(0))/(2 - 0) = (2 - 0)/2 = 1.0$. We find $c \in (0, 2)$ where $f'(c) = 3c^2 - 3 = 1.0 \implies 3c^2 = 4 \implies c = \sqrt{4/3} \approx 1.1547$.

INTERACTIVE LAB

```
import numpy as np
import matplotlib.pyplot as plt

def f(x): return x**3 - 3*x
def df(x): return 3*x**2 - 3

a, b = 0.0, 2.0
secant_slope = (f(b) - f(a)) / (b - a)

# Find c in (a, b) such that df(c) == secant_slope
c = np.sqrt(4.0 / 3.0)

x = np.linspace(-0.2, 2.2, 200)
y = f(x)

plt.plot(x, y, label='f(x) = x^3 - 3x', color='#3b82f6', linewidth=2)
plt.plot([a, b], [f(a), f(b)], '--', color='red', label='Secant Line')
plt.scatter([c], [f(c)], color='black', zorder=5)

# Tangent line at c
tangent_line = f(c) + df(c) * (x - c)
plt.plot(x, tangent_line, ':', color='green', label=f'Tangent at c ≈ {c:.4f}')

plt.legend()
plt.title("Mean Value Theorem Visualization")
plt.grid(True)
plt.show()

print(f"Secant Slope: {secant_slope}")
print(f"Derivative at c: {df(c)}")
```

Critical Corollaries of the MVT

The MVT allows us to transition from local derivative properties to global function properties.

COROLLARY 1.3.3 **Zero Derivative Implies Constant**

Builds on: Theorem 1.3.2

If $f'(x) = 0$ for all x in an open interval I, then f is constant on I.

Pick any two points $x_1, x_2 \in I$ with $x_1 < x_2$. Since f is differentiable on I, it is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). By the MVT, there exists $c \in (x_1, x_2)$ such that:

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since $f'(x) = 0$ for all $x \in I$, we must have $f'(c) = 0$. Therefore,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0 \implies f(x_2) - f(x_1) = 0 \implies f(x_2) = f(x_1).$$

Since this holds for any $x_1, x_2 \in I$, f is constant on I.

∎

REMARK

Even when a derivative f' is discontinuous (such as the derivative of $x^2 \sin(1/x)$ at 0), Darboux's theorem guarantees that f' cannot have simple jump discontinuities: if f is differentiable on $[a, b]$, then f' attains every value between $f'(a)$ and $f'(b)$.

Cauchy's Generalized Mean Value Theorem

We can generalize the MVT to compare two functions parametrically.

THEOREM 1.3.4 **Cauchy's Mean Value Theorem**

Builds on: Theorem 1.3.1

Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $g'(x) \neq 0$ for all $x \in (a, b)$, then there exists some $c \in (a, b)$ such that:

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

First, note $g(b) - g(a) \neq 0$, because if $g(b) = g(a)$, Rolle's Theorem would imply there is a point where $g'(c) = 0$, violating the hypothesis. We define:

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x).$$

1. h is continuous on $[a, b]$ and differentiable on (a, b).
2. Compute $h(a) = f(b)g(a) - g(b)f(a)$ and $h(b) = f(b)g(a) - g(b)f(a)$. Thus $h(a) = h(b)$.
3. By Rolle's Theorem, there exists $c \in (a, b)$ such that $h'(c) = 0$:

$$h'(c) = [f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c) = 0.$$

Dividing by $g'(c)[g(b) - g(a)]$ yields:

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

■

L'Hôpital's Rule

Cauchy's MVT is the fundamental tool used to prove L'Hôpital's Rule for indeterminate limits of the form $0/0$ or ∞/∞.

THEOREM 1.3.5 **L'Hôpital's Rule**

Builds on: Theorem 1.3.4

Let f and g be differentiable on an open interval containing c (except possibly at c). If $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$, and $g'(x) \neq 0$ for $x \neq c$, then:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists.

Worked Trace: Cauchy's Mean Value Theorem Calculation

To see Cauchy's MVT in practice, consider $f(x) = x^2$ and $g(x) = x^3$ on the interval $[1, 2]$.

1. **Verify Hypotheses**: Both f and g are polynomials, so they are continuous on $[1, 2]$ and differentiable on $(1, 2)$. Moreover, $g'(x) = 3x^2 > 0$ for all $x \in (1, 2)$, satisfying $g'(x) \neq 0$.

2. **Compute Secant Quotient**:

$$\frac{f(2) - f(1)}{g(2) - g(1)} = \frac{2^2 - 1^2}{2^3 - 1^3} = \frac{4 - 1}{8 - 1} = \frac{3}{7}.$$

3. **Compute Derivative Ratio**:

$$\frac{f'(c)}{g'(c)} = \frac{2c}{3c^2} = \frac{2}{3c}.$$

4. **Solve for c**:

$$\frac{2}{3c} = \frac{3}{7} \implies 9c = 14 \implies c = \frac{14}{9} \approx 1.5556.$$

5. **Interior Verification**: Since $1 < \frac{14}{9} < 2$, the solution lies strictly in the open interior $(1, 2)$, exactly as guaranteed by the theorem.

Counterexample & Failure Mode: Boundary Breakdown Under Weakened Hypotheses

The Mean Value Theorem requires both continuity on the closed interval $[a, b]$ and differentiability on the open interval (a, b). Violating either condition causes the theorem to collapse immediately.

1. **Failure of Differentiability (Interior Cusp)**: Consider $f(x) = |x|$ on $[-1, 1]$.
   - Continuity holds everywhere on $[-1, 1]$.
   - At $x = 0$, the left-derivative is -1 and right-derivative is $+1$, so f is not differentiable at $x = 0$.
   - Secant slope: $\frac{f(1)-f(-1)}{1-(-1)} = \frac{1-1}{2} = 0$.
   - Instantaneous derivative: $f'(x) = -1$ for $x < 0$, and $f'(x) = +1$ for $x > 0$.
   - **Result**: There is no point $c \in (-1, 1)$ where $f'(c) = 0$. The conclusion fails completely.
2. **Failure of Continuity (Endpoint Jump)**: Consider $f(x) = \begin{cases} 0 & x = 0 \\ 1 & x \in (0, 1] \end{cases}$ on $[0, 1]$.
   - f is differentiable on $(0, 1)$ with $f'(x) = 0$ everywhere.
   - Secant slope: $\frac{f(1)-f(0)}{1-0} = \frac{1-0}{1} = 1$.
   - **Result**: $f'(c) = 0 \ /= 1$ for all $c \in (0, 1)$. The failure of continuity at even a single boundary point invalidates the guarantee.

Concept Check: Boundary Differentiability

KNOWLEDGE CHECK

Consider $f(x) = x^{2/3}$ on $[-1, 1]$ with $f(-1) = f(1) = 1$. Evaluate each statement regarding Rolle's Theorem and the derivative:

Answer: Rolle's Theorem fails to apply because f is not differentiable at the interior point $x = 0$.; No point $c \in (-1, 1)$ satisfies $f'(c) = 0$; the derivative is nonzero wherever it is defined.

While f is continuous on $[-1, 1]$, its derivative $f'(x) = \frac{2}{3}x^{-1/3}$ is undefined at $x = 0$. Rolle's Theorem strictly requires differentiability on the entire open interval (a, b).

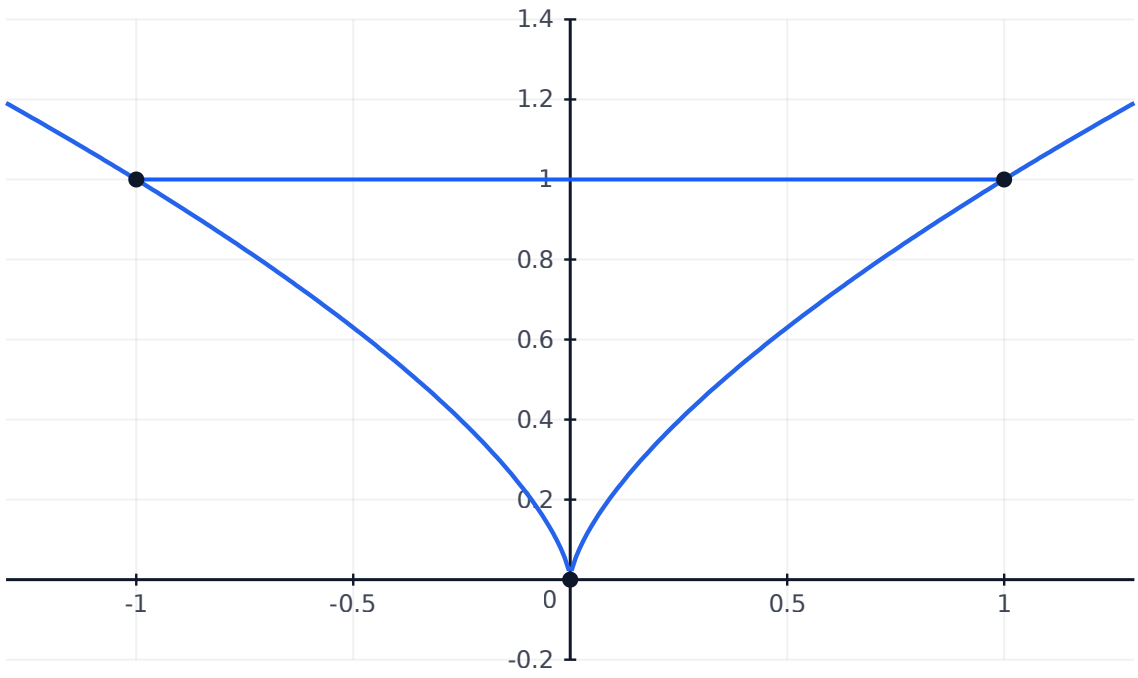


Figure 1.3.3: Failure of the Mean Value Theorem when differentiability fails at an interior point: $f(x) = x^{2/3}$ has chord slope 0 on $[-1, 1]$, but $f'(x) \neq 0$ for all $x \neq 0$.

SECTION EXERCISES

Deep Practice

Apply the Mean Value Theorem and its structural corollaries to establish root bounds, functional inequalities, and indeterminate limits.

Exercise 1.3.1: Rolle's Theorem Root Verification

Verification application

Verify that Theorem 1.3.1 applies to $f(x) = x^3 - 4x$ on the interval $[0, 2]$, and find the exact value of $c \in (0, 2)$ where $f'(c) = 0$.

Check the three hypotheses of Rolle's Theorem: continuity on $[0, 2]$, differentiability on $(0, 2)$, and endpoint equality $f(0) = f(2)$.

1. **Hypothesis 1:** f is a polynomial, hence continuous on $[0, 2]$.
2. **Hypothesis 2:** f is differentiable on $(0, 2)$ with $f'(x) = 3x^2 - 4$.
3. **Hypothesis 3:** $f(0) = 0^3 - 4(0) = 0$, and $f(2) = 2^3 - 4(2) = 8 - 8 = 0$. So $f(0) = f(2) = 0$.
4. **Locating c:** By Theorem 1.3.1, there exists $c \in (0, 2)$ such that $f'(c) = 0$:

$$3c^2 - 4 = 0 \implies c^2 = \frac{4}{3} \implies c = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}.$$

Since $\sqrt{3} \approx 1.732, c \approx 1.155 \in (0, 2)$.

Verification criteria. Requires verifying all three hypotheses and solving for the exact positive root $2/\sqrt{3}$.

Exercise 1.3.2: Mean Value Tangent Point Construction

Construction application

Let $f(x) = x^2$ on $[a, b]$. Find the exact value of $c \in (a, b)$ guaranteed by Theorem 1.3.2. What geometric relationship does c have with the interval endpoints?

Compute the secant slope $\frac{f(b)-f(a)}{b-a}$ and set it equal to $f'(c) = 2c$.

1. The secant slope is:

$$\frac{f(b)-f(a)}{b-a}=\frac{b^2-a^2}{b-a}=\frac{(b-a)(b+a)}{b-a}=b+a=a+b.$$

2. The derivative is $f'(x)=2x$.
3. By Theorem 1.3.2, set $f'(c)=\frac{f(b)-f(a)}{b-a}$:

$$2c=a+b \implies c=\frac{a+b}{2}.$$

4. **Geometric property:** For any quadratic parabola $f(x)=Ax^2+Bx+C$, the tangent line parallel to the secant line occurs *exactly at the midpoint* $c=(a+b)/2$ of the interval.

Verification criteria. A complete derivation requires setting secant slope equal to $2c$ and concluding $c=(a+b)/2$.

Exercise 1.3.3: Boundary Discontinuity Invalidating MVT Guarantee

Example / Impossible transfer

Consider $f(x)=x^{2/3}$ on $[-1,1]$. Does Theorem 1.3.2 guarantee a point $c\in(-1,1)$ where the tangent slope equals the secant slope? Explain why or why not.

Compute the derivative $f'(x)$ and check where it is undefined on the open interval $(-1,1)$.

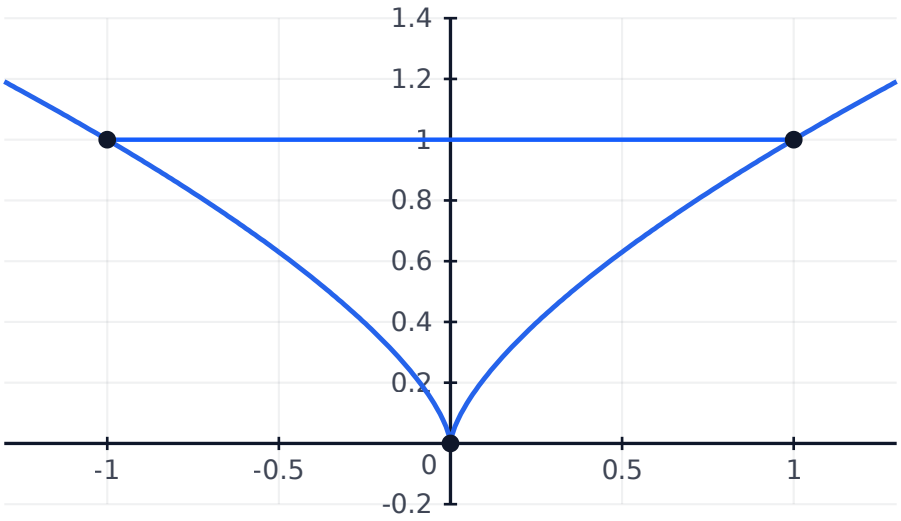


Figure 1.3.3: Failure of the Mean Value Theorem when differentiability fails at an interior point: $f(x) = x^{2/3}$ has chord slope 0 on $[-1, 1]$, but $f'(x) \neq 0$ for all $x \neq 0$.

1. The average rate of change on $[-1, 1]$ is:

$$\frac{f(1) - f(-1)}{1 - (-1)} = \frac{1^{2/3} - (-1)^{2/3}}{2} = \frac{1 - 1}{2} = 0.$$

2. The derivative is:

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}.$$

3. At $x = 0 \in (-1, 1)$, $f'(0)$ is undefined (the graph has a vertical cusp).
4. For all $x \neq 0$, $f'(x) \neq 0$ because the numerator is $2 \neq 0$.
5. Therefore, there is **no point** $c \in (-1, 1)$ with $f'(c) = 0$, even though the secant slope is 0.
6. Theorem 1.3.2 does not apply because f is not differentiable throughout the entire open interval $(-1, 1)$.

Verification criteria. Requires computing secant slope $= 0$, showing f' is undefined at 0 and never zero elsewhere, and explicitly identifying the violation of open-interval differentiability.

Exercise 1.3.4: Root Uniqueness via Strict Monotonicity

Proof transfer

Prove that the polynomial $p(x) = x^5 + 4x - 7$ has **exactly one** real root. State explicitly which structural theorems guarantee (a) existence and (b) uniqueness.

Use Theorem 1.1.5 to establish existence on a compact interval, and Theorem 1.3.1 (or sign of p') to prove uniqueness by contradiction.

Part 1: Existence (via Theorem 1.1.5) $p(x)$ is a polynomial, hence continuous on $\mathbb{R}$. Evaluate at endpoints $x = 1$ and $x = 2$:

$$p(1) = 1^5 + 4(1) - 7 = 1 + 4 - 7 = -2 < 0.$$

$$p(2) = 2^5 + 4(2) - 7 = 32 + 8 - 7 = 33 > 0.$$

Since $p(1) < 0 < p(2)$, by the Intermediate Value Theorem (Theorem 1.1.5), there exists at least one $r \in (1, 2)$ such that $p(r) = 0$.

Part 2: Uniqueness (via Theorem 1.3.1) Suppose for contradiction that $p(x)$ has two distinct real roots $r_1 < r_2$. Then $p(r_1) = p(r_2) = 0$. Since p is continuous on $[r_1, r_2]$ and differentiable on (r_1, r_2), by Rolle's Theorem (Theorem 1.3.1), there must exist some $c \in (r_1, r_2)$ such that $p'(c) = 0$. Now compute the derivative:

$$p'(x) = 5x^4 + 4.$$

Since $x^4 \geq 0$ for all real x, $5x^4 + 4 \geq 4 > 0$ for all $x \in \mathbb{R}$. Thus $p'(x)$ is strictly positive everywhere and can never equal 0. This contradicts the existence of c. Therefore, $p(x)$ cannot have two or more real roots; the root is strictly unique.

Verification criteria. Requires: (a) invoking Theorem 1.1.5 with explicit negative and positive values; (b) assuming two roots for contradiction; (c) invoking Theorem 1.3.1; (d) proving $p'(x) \geq 4 > 0$ to reach contradiction.

Exercise 1.3.5: Proving Functional Inequalities: Sine Bound

Proof synthesis

Use Theorem 1.3.2 to prove that $\sin x < x$ for all $x > 0$.

Apply the Mean Value Theorem to $f(t) = \sin t$ on the interval $[0, x]$ for $x > 0$.

1. Let $x > 0$. Consider $f(t) = \sin t$ on the closed interval $[0, x]$.
2. f is continuous on $[0, x]$ and differentiable on $(0, x)$ with $f'(t) = \cos t$.
3. By Theorem 1.3.2, there exists $c \in (0, x)$ such that:

$$\frac{\sin x - \sin 0}{x - 0} = \cos c \implies \frac{\sin x}{x} = \cos c.$$

4. For any $c \in (0, x)$, we know that $\cos c \leq 1$. Furthermore, since $c \in (0, x)$ and $c > 0$, $\cos c = 1$ can only occur if c is an integer multiple of 2π. But at $c = 0$, $\cos 0 = 1$; for all $c \in (0, 2\pi)$, $\cos c < 1$. In general, for any $c > 0$, the inequality $\cos c < 1$ holds almost everywhere, and specifically on $(0, \min(x, 2\pi))$, $\cos c < 1$. For $x \geq 1$: $\sin x \leq 1 \leq x$, and since $\sin x \leq 1 < x$ for $x > 1$, the claim holds. For $0 < x \leq 1$: $0 < c < x \leq 1 < \pi/2$, where cosine is strictly decreasing from 1, so $\cos c < \cos 0 = 1$.
5. Therefore:

$$\frac{\sin x}{x} = \cos c < 1 \implies \sin x < x.$$

Verification criteria. Requires setting up the interval $[0, x]$, applying Theorem 1.3.2, and justifying the strict inequality $\cos c < 1$.

Exercise 1.3.6: Cauchy MVT Ratio Evaluation

Construction application

Let $f(x) = x^3$ and $g(x) = x^2 + 1$ on $[1, 3]$. Find the exact value of $c \in (1, 3)$ guaranteed by Theorem 1.3.4.

Compute the ratio $\frac{f(3)-f(1)}{g(3)-g(1)}$ and set it equal to $\frac{f'(c)}{g'(c)}$.

1. Compute function values: $f(3) = 27, f(1) = 1 \implies f(3) - f(1) = 26. g(3) = 10, g(1) = 2 \implies g(3) - g(1) = 8.$
2. The secant ratio is:

$$\frac{f(3) - f(1)}{g(3) - g(1)} = \frac{26}{8} = \frac{13}{4}.$$

3. The derivatives are $f'(x) = 3x^2$ and $g'(x) = 2x$. Note $g'(x) = 2x \neq 0$ on $(1, 3)$.
4. By Theorem 1.3.4, set:

$$\frac{f'(c)}{g'(c)} = \frac{3c^2}{2c} = \frac{3c}{2} = \frac{13}{4}.$$

5. Solve for c:

$$3c = \frac{13}{2} \implies c = \frac{13}{6} \approx 2.167.$$

Since $1 < \frac{13}{6} < 3, c \in (1, 3)$ as guaranteed.

Verification criteria. Requires computing both differences, setting $3c/2 = 13/4$, and finding $c = 13/6$.

Exercise 1.3.7: L'Hôpital Evaluation of Quadratic Cosine Indeterminacy

Verification application

Evaluate $\lim_{x \to 0} \frac{1-\cos x}{x^2}$ using Theorem 1.3.5, verifying hypotheses at every step.

SECOND-STEP L'HÔPITAL QUOTIENT

Applying L'Hôpital's Rule once to $\lim_{x \to 0} \frac{1-\cos x}{x^2}$ gives $\lim_{x \to 0} \frac{\sin x}{2x}$, which is still of the indeterminate form $0/0$.
Applying L'Hôpital's Rule a second time yields $\lim_{x \to 0} \frac{\cos x}{2}$ = 1/2

Differentiating numerator and denominator a second time gives $\cos(x)/2$, which evaluates at $x = 0$ to $\cos(0)/2 = 1/2$.

Check that both numerator and denominator evaluate to 0 before each derivative step.

1. **Step 1 (Check initial form):** $\lim_{x \to 0}(1 - \cos x) = 1 - 1 = 0$. $\lim_{x \to 0} x^2 = 0$. Denominator derivative: $\frac{d}{dx}[x^2] = 2x \neq 0$ for $x \neq 0$. The form is indeterminate $0/0$.
2. **Apply Theorem 1.3.5 once:**

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x}.$$

3. **Step 2 (Check second form):** $\lim_{x \to 0} \sin x = 0$ and $\lim_{x \to 0} 2x = 0$. Denominator derivative: $\frac{d}{dx}[2x] = 2 \neq 0$. The form is again $0/0$.
4. **Apply Theorem 1.3.5 a second time:**

$$\lim_{x \to 0} \frac{\sin x}{2x} = \lim_{x \to 0} \frac{\cos x}{2}.$$

5. **Evaluate the direct limit:**

$$\lim_{x \to 0} \frac{\cos x}{2} = \frac{\cos 0}{2} = \frac{1}{2}.$$

   Therefore, the limit is $1/2$.

Verification criteria. A complete derivation requires verifying the $0/0$ condition prior to each differentiation step and obtaining $1/2$.

References & Further Reading

- Spivak, M. (2008). *Calculus* (4th ed.), Chapter 11 ("Significance of the Derivative"). Publish or Perish.
- Rudin, W. (1976). *Principles of Mathematical Analysis* (3rd ed.), Chapter 5 ("Differentiation") — includes Darboux's theorem on intermediate values of derivatives (Theorem 5.12), L'Hôpital's rule (Theorem 5.13), and vector-valued mean value inequalities (Theorem 5.19). McGraw-Hill.

- Apostol, T. M. (1967). *Calculus, Volume 1: One-Variable Calculus, with an Introduction to Linear Algebra* (2nd ed.), Chapter 4 (“The Mean-Value Theorem and Its Applications”). Wiley.

SECTION DETAIL

Problem Set: Limits and Derivatives

Practice problems bridging the first three lessons. These problems demand proof synthesis, hypothesis auditing, and computational tool use across multiple theorems.

★ Problem 1.P.1: Algebra of Limits: Bounding the Denominator

Proof synthesis

Complete the proof of the Algebra of Limits (Theorem 1.1.2): assuming $\lim_{x \to c} g(x) = M \neq 0$, prove there is a $\delta_0 > 0$ such that $|g(x)| > |M|/2$ whenever $0 < |x - c| < \delta_0$; then use this to derive the quotient rule from the product and reciprocal rules.

First pin g away from 0 with $\epsilon = |M|/2$ in the limit definition; the preliminary bound then controls the quotient $|1/g(x) - 1/M|$.

Step 1 (pin away from zero). Apply Definition 1.1.1 with $\epsilon = |M|/2$: there is $\delta_0 > 0$ such that $0 < |x - c| < \delta_0$ implies $|g(x) - M| < |M|/2$. Then by the reverse triangle inequality,

$$|g(x)| = |M + (g(x) - M)| \geq |M| - |g(x) - M| > |M| - \frac{|M|}{2} = \frac{|M|}{2} > 0.$$

Step 2 (reciprocal). For $0 < |x - c| < \delta_0$:

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \frac{|M - g(x)|}{|g(x)|\,|M|} < \frac{|g(x) - M|}{(|M|/2)\,|M|} = \frac{2}{|M|^2}\,|g(x) - M|.$$

Given $\epsilon > 0$, choose δ_1 so that $|g(x) - M| < \epsilon\,|M|^2/2$ for $0 < |x - c| < \delta_1$, and set $\delta = \min(\delta_0, \delta_1)$. Then $|1/g(x) - 1/M| < \epsilon$ throughout, so $\lim_{x \to c} 1/g(x) = 1/M$.

Step 3 (quotient). The product rule of Theorem 1.1.2 applied to f and $1/g$ gives $\lim f/g = L/M$. ∎

Verification criteria. A complete proof requires all three steps: the preliminary pin $|g(x)| > |M|/2$ (with $\epsilon = |M|/2$ explicit), the reciprocal estimate with the $2/|M|^2$ constant appearing, and the final composition with the product rule.

Problem 1.P.2: Numerical Epsilon-Delta Neighborhood Check

Tool / Simulation application

Complete the verifier below so it checks the ϵ-δ condition numerically. Test inputs: $f(x) = 3x - 1, c = 2, L = 5, \epsilon = 0.3, \delta = 0.1$ (must pass), and the same with $\delta = 0.15$ (must fail — the sample points falsify it).

COMPLETE THE EPSILON-DELTA CHECK

```
def verify_limit(f, c, L, epsilon, delta, samples=1000, seed=0):
  # Check random points in the delta-neighborhood of c (excluding c
itself)
  import numpy as np
  rng = np.random.default_rng(seed)
  x_vals = c + rng.uniform(-delta, delta, samples)
  x_vals = x_vals[x_vals != c]

  # The condition is: 0 < |x - c| < delta implies |f(x) - L| < epsilon
  satisfied = np.all(np.abs(f(x_vals) - L) < epsilon)
  return satisfied
```

The condition requires that the absolute difference between the function value and the limit is strictly less than epsilon for every sampled point. With the test inputs above: for δ=0.1 every sampled $|f(x)-5| = 3|x-2| < 0.3$ holds; for δ=0.15 there are sampled points with $|x-2| > 0.1$, where $|f(x)-5| > 0.3$, so the verifier correctly returns False.

Inspect the vectorized evaluation in NumPy: the test asserts that all points in the punctured neighborhood satisfy the epsilon inequality.

The blank must compute the elementwise absolute deviation `np.abs(f(x_vals) - L)`. Passing $\delta = 0.1$ verifies that for all 1000 points $|f(x) - 5| = 3|x - 2| < 3(0.1) = 0.3$. For $\delta = 0.15$, points near the boundaries $x = 1.86$ or $x = 2.14$ yield deviations $|f(x) - 5| \approx 0.42 > 0.3$, causing `np.all` to return `False`.

★ Problem 1.P.3: Quantifier Perturbation: The 'Verconges' Definition

Perturbed Definition | transfer

Suppose a mathematician proposes an altered definition of limit: a function f is said to *verconge* to L at c if: **There exists $\delta > 0$ such that for every $\epsilon > 0$**, for all x:

$$0 < |x - c| < \delta \implies |f(x) - L| < \epsilon.$$

Characterize all functions that verconge to L at c. What does this reveal about why the order of quantifiers in Definition 1.1.1 matters?

If $|f(x) - L| < \epsilon$ for every $\epsilon > 0$ on a fixed interval, what is the only non-negative number that can satisfy that?

Characterization: A function f verconges to L at c **if and only if** $f(x) = L$ **identically on the entire punctured neighborhood** $(c-\delta, c+\delta) \setminus \{c\}$.

Proof: Suppose f verconges to L at c, so there exists a fixed $\delta > 0$ such that for every $x \in (c-\delta, c+\delta) \setminus \{c\}$, the distance $|f(x) - L| < \epsilon$ for *every* $\epsilon > 0$. The only non-negative real number that is strictly less than every positive real number ϵ is 0. Therefore, for every x in the punctured window, $|f(x) - L| = 0$, which implies $f(x) = L$. Conversely, if $f(x) = L$ on $(c-\delta, c+\delta) \setminus \{c\}$, then $|f(x) - L| = 0 < \epsilon$ holds for all $\epsilon > 0$.

Significance: In Definition 1.1.1, δ is chosen *after* ϵ ($\forall \epsilon > 0, \exists \delta > 0$). This allows δ to shrink as the tolerance ϵ tightens, capturing *local sensitivity* and convergence. Swapping the quantifiers forces one static window δ to satisfy all tolerances at once, which destroys the entire theory of continuous change and restricts the concept to trivial locally constant functions.

Problem 1.P.4: Epsilon-Delta Derivative Existence and Lipschitz Bound

Proof synthesis

Let $f(x) = x^3$.

1. Using Definition 1.1.1, give a rigorous ϵ-δ proof that $\lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = 3x^2$ at an arbitrary fixed $x \in \mathbb{R}$.
2. Using Theorem 1.3.2, show that for all $x_1, x_2 \in [1, 3]$, $|x_1^3 - x_2^3| \leq 27|x_1 - x_2|$.

Expand $(x+h)^3$, factor out h, and freeze the variable factor in the difference quotient with a preliminary bound $|h| \leq 1$.

Part 1: ϵ-δ proof of the derivative. For $h \neq 0$:

$$\frac{(x+h)^3 - x^3}{h} = \frac{3x^2h + 3xh^2 + h^3}{h} = 3x^2 + 3xh + h^2.$$

The difference from the candidate limit $3x^2$ is:

$$\left|\left(3x^2 + 3xh + h^2\right) - 3x^2\right| = |3xh + h^2| = |h|\,|3x + h|.$$

Impose the preliminary bound $|h| \leq 1$. Then by the triangle inequality, $|3x + h| \leq 3|x| + |h| \leq 3|x| + 1$. Given $\epsilon > 0$, choose:

$$\delta = \min\left(1, \frac{\epsilon}{3|x| + 1}\right).$$

Whenever $0 < |h| < \delta$, we have $|h| \leq 1$ and:

$$\left|\frac{(x+h)^3 - x^3}{h} - 3x^2\right| \leq |h|(3|x| + 1) < \frac{\epsilon}{3|x| + 1}(3|x| + 1) = \epsilon.$$

This establishes that $f'(x) = 3x^2$ directly from Definition 1.1.1.

Part 2: Lipschitz bound via MVT. Let $x_1, x_2 \in [1, 3]$ with $x_1 < x_2$. By Theorem 1.3.2 applied to $f(t) = t^3$ on $[x_1, x_2]$, there exists $c \in (x_1, x_2)$ such that:

$$x_2^3 - x_1^3 = f'(c)(x_2 - x_1) = 3c^2(x_2 - x_1).$$

Since $c \in (x_1, x_2) \subset [1, 3]$, $c \leq 3$, so $3c^2 \leq 3(3^2) = 27$. Taking absolute values:

$$|x_2^3 - x_1^3| = 3c^2|x_2 - x_1| \leq 27|x_2 - x_1|.$$

★ Problem 1.P.5: Differentiable Function with Discontinuous Derivative

Example / Impossible transfer

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 \sin(1/x)$ for $x \neq 0$, and $f(0) = 0$. Prove that f is differentiable everywhere on $\mathbb{R}$, but the derivative function f' is discontinuous at $x = 0$.

Use the limit definition directly at $x = 0$. For $x \neq 0$, use standard differentiation rules and inspect the limit of $f'(x)$ as $x \to 0$.

Differentiability at 0: By Definition 1.2.1:

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin(1/h) - 0}{h} = \lim_{h \to 0} h \sin(1/h).$$

Since $|\sin(1/h)| \leq 1$, we have $|h \sin(1/h)| \leq |h| \to 0$. By the Squeeze Theorem, $f'(0) = 0$. Thus f is differentiable at 0.

Derivative for $x \neq 0$: By Theorem 1.2.3 and Theorem 1.2.4:

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) + x^2 \cos\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right).$$

Discontinuity of f' at 0: As $x \to 0, 2x \sin(1/x) \to 0$, but $-\cos(1/x)$ oscillates between -1 and 1 in every neighborhood of 0. Therefore, $\lim_{x \to 0} f'(x)$ does not exist. Since $\lim_{x \to 0} f'(x) \neq f'(0) = 0$, f' is discontinuous at 0.

Conclusion: Differentiability guarantees f is continuous (Theorem 1.2.2), but does **not** guarantee f' is continuous.

★ Problem 1.P.6: The False Proof and Flaw in the Chain Rule

Proof synthesis

A proposed derivation offers the following proof of Theorem 1.2.4:

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h}.$$

Taking $h \to 0$, the first term approaches $f'(g(x))$ and the second approaches $g'(x)$, giving $f'(g(x))g'(x)$. Diagnose the logical flaw in this argument. How does Carathéodory's definition or an auxiliary function resolve it?

What happens if $g(x+h) = g(x)$ for values of h arbitrarily close to 0? Can we divide by zero in the difference quotient?

The Flaw: The proof divides by $g(x+h)-g(x)$. However, g may be constant in a neighborhood of x, or worse, oscillate infinitely often such that $g(x+h)=g(x)$ for points $h_n \to 0$ (for instance, $g(t)=x+t^2\sin(1/t)$). For such h, the fraction $\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}$ is undefined $(0/0)$. One cannot divide by a quantity that may vanish along the limit sequence.

The Resolution (Carathéodory's approach): Define the auxiliary difference-quotient slope function Φ on the domain of f:

$$\Phi(y)=\begin{cases}\frac{f(y)-f(u)}{y-u} & \text{if } y \neq u \\ f'(u) & \text{if } y=u\end{cases}$$

where $u=g(x)$. Because f is differentiable at u, $\lim_{y\to u}\Phi(y)=f'(u)=\Phi(u)$, so Φ is **continuous** at u. Notice that $f(y)-f(u)=\Phi(y)(y-u)$ holds unconditionally for **all** y, even when $y=u$! Now substitute $y=g(x+h)$:

$$\frac{f(g(x+h))-f(g(x))}{h}=\Phi(g(x+h))\cdot\frac{g(x+h)-g(x)}{h}.$$

Since g is differentiable at x, it is continuous at x, so $\lim_{h\to 0} g(x+h)=g(x)=u$. Because Φ is continuous at u, $\lim_{h\to 0}\Phi(g(x+h))=\Phi(u)=f'(g(x))$. Multiplying limits:

$$(f\circ g)'(x)=f'(g(x))\cdot g'(x).$$

This proof never divides by $g(x+h)-g(x)$ and works unconditionally for all differentiable functions.

Problem 1.P.7: Central Difference Truncation Error Analysis

Tool / Simulation application

Suppose f is three times differentiable on an open interval containing x with f''' continuous at x. Complete the numerical central difference formula below and prove using Theorem 1.3.5 that its truncation error satisfies

$$\lim_{h \to 0} \frac{\frac{f(x+h) - f(x-h)}{2h} - f'(x)}{h^2} = \frac{f'''(x)}{6},$$

establishing that the central difference error is $\mathcal{O}(h^2)$ rather than $\mathcal{O}(h)$.

COMPLETE THE NUMERICAL DERIVATIVE (CENTRAL DIFFERENCE)

```
def central_difference(f, x, h=1e-5):
  # Central difference formula provides O(h^2) accuracy compared to
forward O(h)
  return (f(x + h) - f(x - h)) / (2 * h)
```

The central difference formula is (f(x + h) - f(x - h)) / (2h), which cancels the first-order error term and leaves O(h²) truncation error.

Rewrite the ratio as $\frac{f(x+h) - f(x-h) - 2hf'(x)}{2h^3}$ and apply L'Hôpital's Rule with respect to h three times.

Derivation of Truncation Error via L'Hôpital's Rule: Combine the numerator over the common denominator $2h^3$:

$$E(h) = \frac{\frac{f(x+h)-f(x-h)}{2h} - f'(x)}{h^2} = \frac{f(x+h) - f(x-h) - 2hf'(x)}{2h^3}.$$

1. **First application of** Theorem 1.3.5**:** As $h \to 0$, both numerator and denominator approach 0. Differentiating with respect to h (using Theorem 1.2.4 on $f(x-h)$):

$$\lim_{h\to 0} \frac{f(x+h) - f(x-h) - 2hf'(x)}{2h^3} = \lim_{h\to 0} \frac{f'(x+h) + f'(x-h) - 2f'}{6h^2}$$

2. **Second application of** Theorem 1.3.5**:** At $h = 0$, the numerator is $f'(x) + f'(x) - 2f'(x) = 0$ and the denominator is 0. Differentiating with respect to h again:

$$\lim_{h\to 0} \frac{f''(x+h) - f''(x-h)}{12h}.$$

3. **Evaluate via** Definition 1.2.1 **(or a third L'Hôpital step):**

$$\lim_{h\to 0} \frac{1}{6} \cdot \frac{f''(x+h) - f''(x-h)}{2h} = \frac{1}{6} f'''(x).$$

Thus $\frac{f(x+h)-f(x-h)}{2h} - f'(x) = \frac{f'''(x)}{6}h^2 + o(h^2) = \mathcal{O}(h^2)$, whereas the one-sided forward difference $\frac{f(x+h)-f(x)}{h} - f'(x)$ has first-order error $\frac{f''(x)}{2}h + o(h) = \mathcal{O}(h)$.

Problem 1.P.8: Carathéodory Equivalence Formulation

Formalisation transfer

Carathéodory's Theorem states: f is differentiable at c if and only if there exists a function ϕ continuous at c such that $f(x) - f(c) = \phi(x)(x - c)$ for all x. Prove that if such a ϕ exists, then $f'(c) = \phi(c)$.

Take the limit as $x \to c$ of $(f(x) - f(c))/(x - c)$ and use the continuity of ϕ.

Proof: Suppose there exists a function ϕ continuous at c such that:

$$f(x) - f(c) = \phi(x)(x - c) \quad \text{for all } x.$$

For any $x \neq c$, we can divide by $x - c$:

$$\frac{f(x) - f(c)}{x - c} = \phi(x).$$

Now evaluate the limit as $x \to c$ using Definition 1.2.1:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \phi(x).$$

Because ϕ is assumed continuous at c, Definition 1.1.4 guarantees that $\lim_{x \to c} \phi(x) = \phi(c)$. Therefore, the limit exists and:

$$f'(c) = \phi(c).$$

Conversely, if f is differentiable at c, defining $\phi(x) = \frac{f(x) - f(c)}{x - c}$ for $x \neq c$ and $\phi(c) = f'(c)$ yields a function continuous at c that satisfies the equation for all x. ∎

Problem 1.P.9: Second-Order Remainder Bound via Cauchy Mean Value Theorem

Construction synthesis

Let f be twice differentiable on $[a, b]$, and let $L(x) = f(a) + f'(a)(x - a)$ be the linear approximation at a. Prove that for any $x \in (a, b]$, there exists $c \in (a, x)$ such that:

$$f(x) - L(x) = \frac{f''(c)}{2}(x - a)^2.$$

Apply Cauchy's Mean Value Theorem to $F(t) = f(x) - f(t) - f'(t)(x - t)$ and $G(t) = (x - t)^2$ on $[a, x]$.

Define two auxiliary functions on $[a, x]$:

$$F(t) = f(x) - f(t) - f'(t)(x - t),$$

$$G(t) = (x - t)^2.$$

Evaluate at $t = x$: $F(x) = f(x) - f(x) - f'(x)(0) = 0$, and $G(x) = (x - x)^2 = 0$. Evaluate at $t = a$: $F(a) = f(x) - f(a) - f'(a)(x - a) = f(x) - L(x)$, and $G(a) = (x - a)^2$. Now differentiate with respect to t:

$$F'(t) = -f'(t) - [f''(t)(x - t) - f'(t)] = -f''(t)(x - t),$$

$$G'(t) = -2(x - t).$$

Note that for $t \in (a, x)$, $G'(t) = -2(x - t) \ /= 0$. By Cauchy's Mean Value Theorem (Theorem 1.3.4), there exists $c \in (a, x)$ such that:

$$\frac{F(x) - F(a)}{G(x) - G(a)} = \frac{F'(c)}{G'(c)}.$$

Substituting the values:

$$\frac{0 - (f(x) - L(x))}{0 - (x - a)^2} = \frac{-f''(c)(x - c)}{-2(x - c)} = \frac{f''(c)}{2}.$$

Multiplying by $(x - a)^2$:

$$f(x) - L(x) = \frac{f''(c)}{2}(x - a)^2.$$

This establishes Lagrange's form of the Taylor remainder for degree 1.

★ Problem 1.P.10: Circular Reasoning in Differentiation of Sine

Negation transfer

Explain why using Theorem 1.3.5 to evaluate $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$ is circular reasoning when defining the derivative of sine.

How is the derivative of $\sin x$ proved in the first place? Inspect the difference quotient of sine.

The Circularity: To apply Theorem 1.3.5 to $\lim_{x\to 0} \frac{\sin x}{x}$, one must differentiate $\sin x$ at $x = 0$ to get $\cos 0$. However, by Definition 1.2.1, the derivative of $\sin x$ at 0 is defined as:

$$\sin'(0) = \lim_{h\to 0} \frac{\sin(0+h) - \sin 0}{h} = \lim_{h\to 0} \frac{\sin h}{h}.$$

Thus, knowing that $\frac{d}{dx}[\sin x] = \cos x$ **already requires knowing that** $\lim_{h\to 0} \frac{\sin h}{h} = 1$! Using L'Hôpital's Rule to prove that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ assumes the very truth of the fact being evaluated. The limit must be proved independently using geometry and the Squeeze Theorem (Theorem 1.1.3).

Problem 1.P.11: Failure of Equality in Vector-Valued MVT

Description transfer

Let $\mathbf{f} : [0, 2\pi] \to \mathbb{R}^2$ be given by $\mathbf{f}(t) = (\cos t, \sin t)$. Explain why there is no point $c \in (0, 2\pi)$ satisfying $\mathbf{f}'(c) = \frac{\mathbf{f}(2\pi) - \mathbf{f}(0)}{2\pi - 0}$. What inequality replaces the MVT in higher dimensions?

Compute the secant vector and the velocity vector $\|\mathbf{f}'(t)\|$.

1. The secant vector is:

$$\frac{\mathbf{f}(2\pi) - \mathbf{f}(0)}{2\pi} = \frac{(1, 0) - (1, 0)}{2\pi} = (0, 0).$$

2. The derivative vector is $\mathbf{f}'(t) = (-\sin t, \cos t)$.
3. Its Euclidean norm is:

$$\|\mathbf{f}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2} = \sqrt{1} = 1 \quad \text{for all } t \in [0, 2\pi].$$

4. Since the velocity vector is everywhere of length 1, it is never equal to the zero vector $(0, 0)$. Thus no such c exists.
5. In vector and multivariable analysis, equality is replaced by the **Mean Value Inequality**:

$$\|\mathbf{f}(b) - \mathbf{f}(a)\| \leq (b - a) \sup_{c \in (a,b)} \|\mathbf{f}'(c)\|.$$

★ Problem 1.P.12: The Open Interval Hypothesis and Disconnected Domains

Perturbed Definition | synthesis

In Corollary 1.3.3, the hypothesis states that $f'(x) = 0$ on an **open interval** I. Construct an example of a function f whose derivative is identically zero everywhere on its domain, but f is **not** constant. Why does this not contradict Corollary 1.3.3?

Consider a domain that consists of two disjoint intervals, such as $(0, 1) \cup (2, 3)$.

Construction: Let the domain be $D = (0, 1) \cup (2, 3)$. Define $f : D \to \mathbb{R}$ by:

$$f(x) = \begin{cases} 5 & \text{if } x \in (0, 1) \\ 9 & \text{if } x \in (2, 3) \end{cases}$$

1. On $(0, 1)$, f is constant, so $f'(x) = 0$.
2. On $(2, 3)$, f is constant, so $f'(x) = 0$. Thus $f'(x) = 0$ for **all** $x \in D$. However, f is not constant on D: $f(0.5) = 5 \ /= 9 = f(2.5)$.

Why Corollary 1.3.3 is not contradicted: Corollary 1.3.3 requires the domain to be a **single connected interval** I. If $x1 \in (0, 1)$ and $x2 \in (2, 3)$, the line segment $[x1, x2]$ is not contained in D (it contains the gap $[1, 2]$ where f is undefined). Because the domain is disconnected, Theorem 1.3.2 cannot be applied across the gap. The proof of Corollary 1.3.3 strictly requires $[x1, x2] \subseteq D$.

Problem 1.P.13: Rigorous Proof of L'Hôpital's Rule for the 0/0 Form

Proof synthesis

Assume f and g are differentiable on (c, b) with $g'(x) \neq 0$, and $\lim_{x \to c^+} f(x) = \lim_{x \to c^+} g(x) = 0$. Assume further that $\lim_{x \to c^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$. Prove that $\lim_{x \to c^+} \frac{f(x)}{g(x)} = L$.

Extend f and g continuously to c by defining $f(c) = g(c) = 0$, then apply Cauchy's Mean Value Theorem on $[c, x]$.

Extend f and g to $[c, b)$ by defining $f(c) = 0$ and $g(c) = 0$. Since $\lim_{x \to c^+} f(x) = 0 = f(c)$ and $\lim_{x \to c^+} g(x) = 0 = g(c)$, both functions are continuous on $[c, x]$ for every $x \in (c, b)$. Both functions are differentiable on (c, x), and $g'(t) \neq 0$ on (c, x). By Rolle's Theorem, $g(x) - g(c) = g(x) \neq 0$ (otherwise g' would vanish). By Cauchy's Mean Value Theorem (Theorem 1.3.4), there exists some $\xi_x \in (c, x)$ such that:

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(c)}{g(x) - g(c)} = \frac{f'(\xi_x)}{g'(\xi_x)}.$$

Now consider the limit as $x \to c^+$. Since $c < \xi_x < x$, the Squeeze Theorem (Theorem 1.1.3) forces $\lim_{x \to c^+} \xi_x = c^+$. Therefore:

$$\lim_{x \to c^+} \frac{f(x)}{g(x)} = \lim_{x \to c^+} \frac{f'(\xi_x)}{g'(\xi_x)} = \lim_{\xi \to c^+} \frac{f'(\xi)}{g'(\xi)} = L.$$

This completes the proof of Theorem 1.3.5 for the right-sided $0/0$ form. ∎

Problem 1.P.14: Root Interlacing Between Function and Derivative

Proof transfer

Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable.

1. Prove that between any two real roots of f, there exists at least one real root of f'.
2. Deduce that if $f''(x) > 0$ for all $x \in \mathbb{R}$, then f can have at most two real roots.

Apply Rolle's Theorem to consecutive roots of f, then examine monotonicity of f' from the sign of f''.

Part 1: Interlacing of roots. Let $r1 < r2$ be two distinct real roots of f, so $f(r1) = f(r2) = 0$. Since f is differentiable on $\mathbb{R}$, it is continuous on $[r1, r2]$ and differentiable on $(r1, r2)$. By Rolle's Theorem (Theorem 1.3.1), there exists at least one $c \in (r1, r2)$ such that $f'(c) = 0$.

Part 2: Bound on number of roots. Suppose for contradiction that f has three or more distinct roots $r1 < r2 < r3$. By Part 1, there exists $c1 \in (r1, r2)$ with $f'(c1) = 0$, and there exists $c2 \in (r2, r3)$ with $f'(c2) = 0$. Since $r1 < c1 < r2 < c2 < r3$, we have $c1 < c2$. Now apply Rolle's Theorem to f' on $[c1, c2]$: since f' is continuous on $[c1, c2]$ and differentiable on $(c1, c2)$ with $f'(c1) = f'(c2) = 0$, there exists $\xi \in (c1, c2)$ such that:

$$f''(\xi) = 0.$$

However, the hypothesis states that $f''(x) > 0$ for all $x \in \mathbb{R}$. This contradiction proves that f cannot have three or more roots; it can have at most two real roots.

References & Further Reading

- Michael Spivak (2008). *Calculus* (4th ed.), Chapters 5, 9, 11. Publish or Perish.
- Walter Rudin (1976). *Principles of Mathematical Analysis* (3rd ed.), Chapters 4 and 5. McGraw-Hill.
- Stephen Abbott (2015). *Understanding Analysis* (2nd ed.), Chapters 4 and 5. Springer.

SECTION DETAIL

Limits and derivatives recap

This recap sheet synthesises the foundational results of differential calculus developed across Chapter 1. It consolidates formal definitions, core theorems, decision criteria for method selection, critical failure modes when hypotheses fail, and pervasive operational errors.

Definitions and Results

1. Limits and Continuity

- **Formal Limit Definition** (Definition 1.1.1): For $f : D \to \mathbb{R}$ and limit point c, $\lim_{x \to c} f(x) = L$ if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$. The punctured neighbourhood guarantees that $f(c)$ need not exist nor equal L.
- **Algebra of Limits** (Theorem 1.1.2): If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, limits commute linearly with addition ($\lim(f + g) = L + M$), scalar multiplication ($\lim(kf) = kL$), product ($\lim(fg) = LM$), and quotient ($\lim(f/g) = L/M$, provided $M \neq 0$).
- **Squeeze (Sandwich) Theorem** (Theorem 1.1.3): If $g(x) \leq f(x) \leq h(x)$ on a punctured neighbourhood of c and $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$, then $\lim_{x \to c} f(x) = L$. Used to bound oscillating functions such as $x^2 \sin(1/x)$ between envelopes $\pm x^2$.

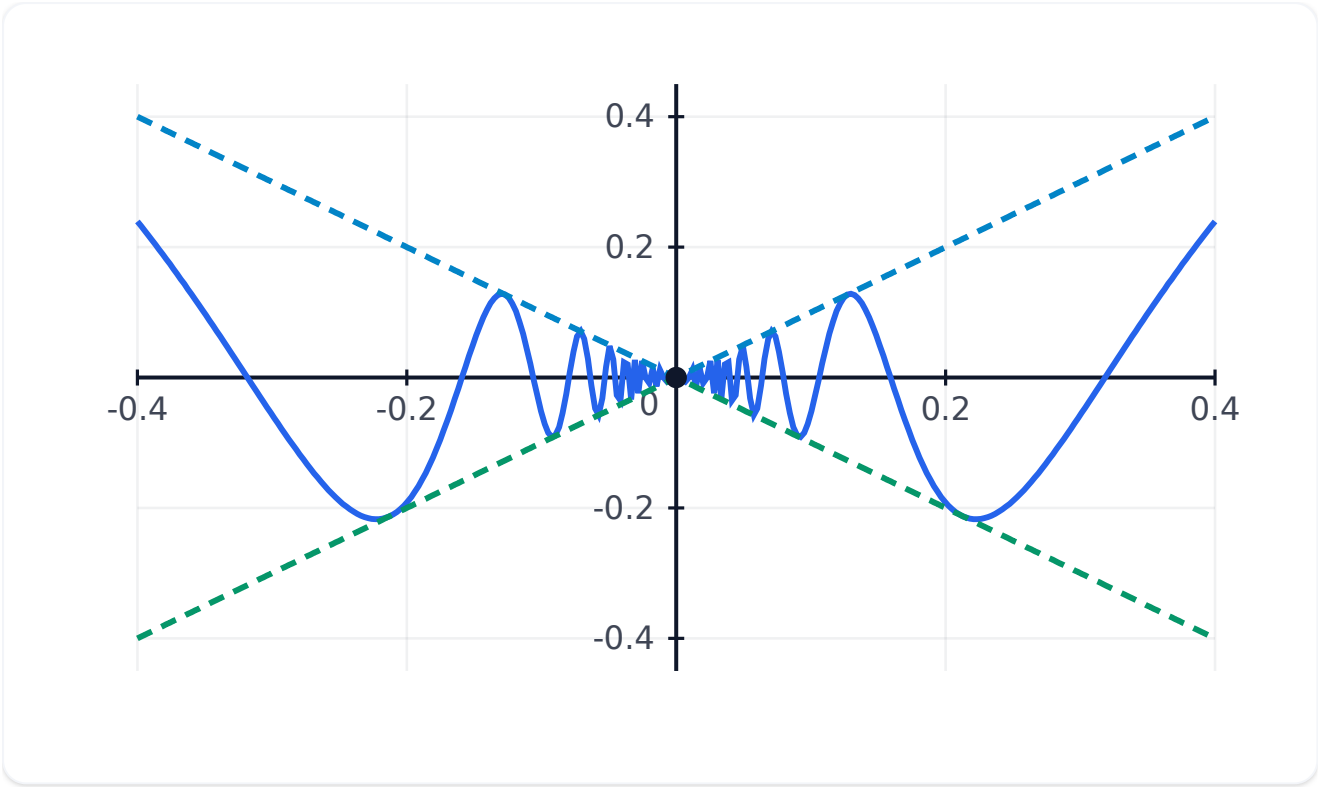


Figure 1.1.2: The squeeze theorem: $-|x| \le x\sin(1/x) \le |x|$, trapping the oscillating function at the origin.

- **Continuity** (Definition 1.1.4): A function f is continuous at $c \in D$ if $\lim_{x \to c} f(x) = f(c)$. Graphically, no jumps, holes, or essential oscillations occur at c.
- **Intermediate Value Theorem (IVT)** (Theorem 1.1.5): If $f \in C([a, b])$ and u lies strictly between $f(a)$ and $f(b)$, there exists at least one $c \in (a, b)$ such that $f(c) = u$. Guarantees roots when $f(a)f(b) < 0$.
- **Extreme Value Theorem (EVT)** (Theorem 1.1.6): If $f \in C([a, b])$ on a closed, bounded interval, then f is bounded and attains both an absolute maximum and an absolute minimum on $[a, b]$.

2. The Derivative and Differentiation Rules

- **Derivative Definition** (Definition 1.2.1): The instantaneous rate of change is defined by $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{z \to x} \frac{f(z)-f(x)}{z-x}$, whenever this limit exists. Geometrically, it is the slope of the unique tangent line to $y = f(x)$ at $(x, f(x))$.
- **Differentiability Implies Continuity** (Theorem 1.2.2): If $f'(c)$ exists, then f is continuous at c. The converse fails: continuity does not imply differentiability (e.g., sharp corners, cusps, or infinite oscillations).
- **Linearity and Product/Quotient Rules** (Theorem 1.2.3): Differentiation is linear: $(af + bg)' = af' + bg'$. The Leibniz product rule gives $(fg)' = f'g + fg'$, and the quotient rule gives $(f/g)' = \frac{f'g - fg'}{g^2}$ where $g(x) \neq 0$.

- **Chain Rule** (Theorem 1.2.4): If g is differentiable at x and f is differentiable at $g(x)$, the composition $h = f \circ g$ is differentiable at x with $h'(x) = f'(g(x)) \cdot g'(x)$.

3. Mean Value Theorems and Asymptotic Analysis

- **Rolle's Theorem** (Theorem 1.3.1): If $f \in C([a, b])$, f is differentiable on (a, b), and $f(a) = f(b)$, there exists at least one $c \in (a, b)$ such that $f'(c) = 0$.
- **Mean Value Theorem (MVT)** (Theorem 1.3.2): If $f \in C([a, b])$ and f is differentiable on (a, b), there exists $c \in (a, b)$ satisfying $f'(c) = \frac{f(b)-f(a)}{b-a}$. The instantaneous rate matches the average rate over the interval.
- **Cauchy's Extended Mean Value Theorem** (Theorem 1.3.4): If $f, g \in C([a, b])$ are differentiable on (a, b) with $g'(x) \ /= 0$, there exists $c \in (a, b)$ such that $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$.
- **Zero-Derivative Characterisation** (Corollary 1.3.3): If $f'(x) = 0$ for all x in an open interval I, then f is constant on I. Consequently, two functions with identical derivatives on I differ by a constant.
- **L'Hôpital's Rule** (Theorem 1.3.5): For indeterminate forms of type $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, if $\lim \frac{f'(x)}{g'(x)} = L \in \mathbb{R} \cup \{\pm\infty\}$ and $g'(x) \ /= 0$ near c, then $\lim_{x \to c} \frac{f(x)}{g(x)} = L$.

Method Chooser

Problem Shape / Goal	Recommended Tool	Step-by-Step Execution
Indeterminate form $\frac{0}{0}$ (algebraic polynomials)	Factorisation / Conjugate Multiplication	Factor out $(x-c)$ or multiply by the conjugate radical $\sqrt{P(x)}+\sqrt{Q(x)}$ to cancel vanishing denominators before evaluating.
Indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ (transcendental functions)	Theorem 1.3.5	Verify that both numerator and denominator simultaneously vanish or diverge. Differentiate numerator and denominator separately until the form becomes determinate.
Oscillating bounded factor (e.g., $\sin(1/x)$ or $(-1)^n$)	Theorem 1.1.3	Establish rigorous upper and lower bounds $g(x) \le f(x) \le h(x)$ where $\lim g(x) = \lim h(x) = L$.
Prove existence of at least one root $f(x)=0$	Theorem 1.1.5	Verify $f \in C([a,b])$. Find points a, b with $f(a)f(b) < 0$. Conclude existence of $c \in (a,b)$ with $f(c)=0$.
Prove existence of a unique root	Theorem 1.1.5 + Theorem 1.3.1	Use IVT for existence. For uniqueness, assume two roots $x_1 < x_2$; by Rolle's theorem, $f'(c)=0$ for some $c \in (x_1, x_2)$. If $f'(x) > 0$ or $f'(x) < 0$ strictly, this yields a contradiction.
Bound error / prove global inequalities $\lvert f(b)-f(a)\rvert \le M\lvert b-a\rvert$	Theorem 1.3.2	Express $f(b)-f(a) = f'(c)(b-a)$ for some $c \in (a,b)$. Apply uniform bounds $\sup_x \lvert f'(x)\rvert \le M$ to obtain Lipschitz bounds.
Differentiate nested functions $f(g(h(x)))$	Theorem 1.2.4	Decompose into composition layers. Work strictly from the outermost derivative to innermost: $f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$.

Hypothesis Checklist

Every hypothesis in analysis is sharp. Omitting even one condition causes the conclusion to fail.

- **Extreme Value Theorem (Theorem 1.1.6)**
  - *Missing Closed Interval:* Let $f(x) = \frac{1}{x}$ on $(0, 1]$. Continuous on $(0, 1]$, but unbounded as $x \to 0^+$; no maximum exists. For a bounded example, $f(x) = x$ on $(0, 1)$ has $\sup f = 1$ and $\inf f = 0$, but attains neither.
  - *Missing Bounded Interval:* Let $f(x) = x$ on $[0, \infty)$. Continuous and closed, but unbounded; no maximum exists.
  - *Missing Continuity:* Let $f(x) = x$ for $x \in [0, 1)$ and $f(1) = 0$. Defined on compact $[0, 1]$, but discontinuous at $x = 1$. The supremum is 1, but $f(x) < 1$ for all $x \in [0, 1]$; no maximum is attained.
- **Intermediate Value Theorem (Theorem 1.1.5)**
  - *Missing Continuity:* Let $f(x) = \operatorname{sgn}(x)$ on $[-1, 1]$ (with $f(0) = 1$). $f(-1) = -1 < 0$ and $f(1) = 1 > 0$, but $f(x)$ never equals 0.
- **Zero-Derivative Characterisation (Corollary 1.3.3)**
  - *Missing Connected Domain:* Let $D = (0, 1) \cup (2, 3)$. Define $f(x) = 0$ on $(0, 1)$ and $f(x) = 5$ on $(2, 3)$. Then $f'(x) = 0$ for every $x \in D$, yet f is not constant on D. Differentiability implies constancy only on connected components (intervals).
- **Mean Value Theorem (Theorem 1.3.2)**
  - *Missing Continuity on $[a, b]$ (Endpoint discontinuity):* Let $f(x) = 0$ for $x \in [0, 1)$ and $f(1) = 1$. f is differentiable on $(0, 1)$ with $f'(x) = 0$. The secant slope is $\frac{1-0}{1-0} = 1$. No point in $(0, 1)$ has $f'(c) = 1$.
  - *Missing Differentiability on (a, b):* Let $f(x) = x^{2/3}$ on $[-1, 1]$ (see the cusp counterexample below). Then $f \in C([-1, 1])$ and the secant slope is $\frac{f(1)-f(-1)}{1-(-1)} = \frac{1-1}{2} = 0$. However, $f'(0)$ fails to exist at the vertical cusp $x = 0$, and $f'(x) = \frac{2}{3}x^{-1/3} \neq 0$ for all $x \neq 0$; no $c \in (-1, 1)$ achieves $f'(c) = 0$.

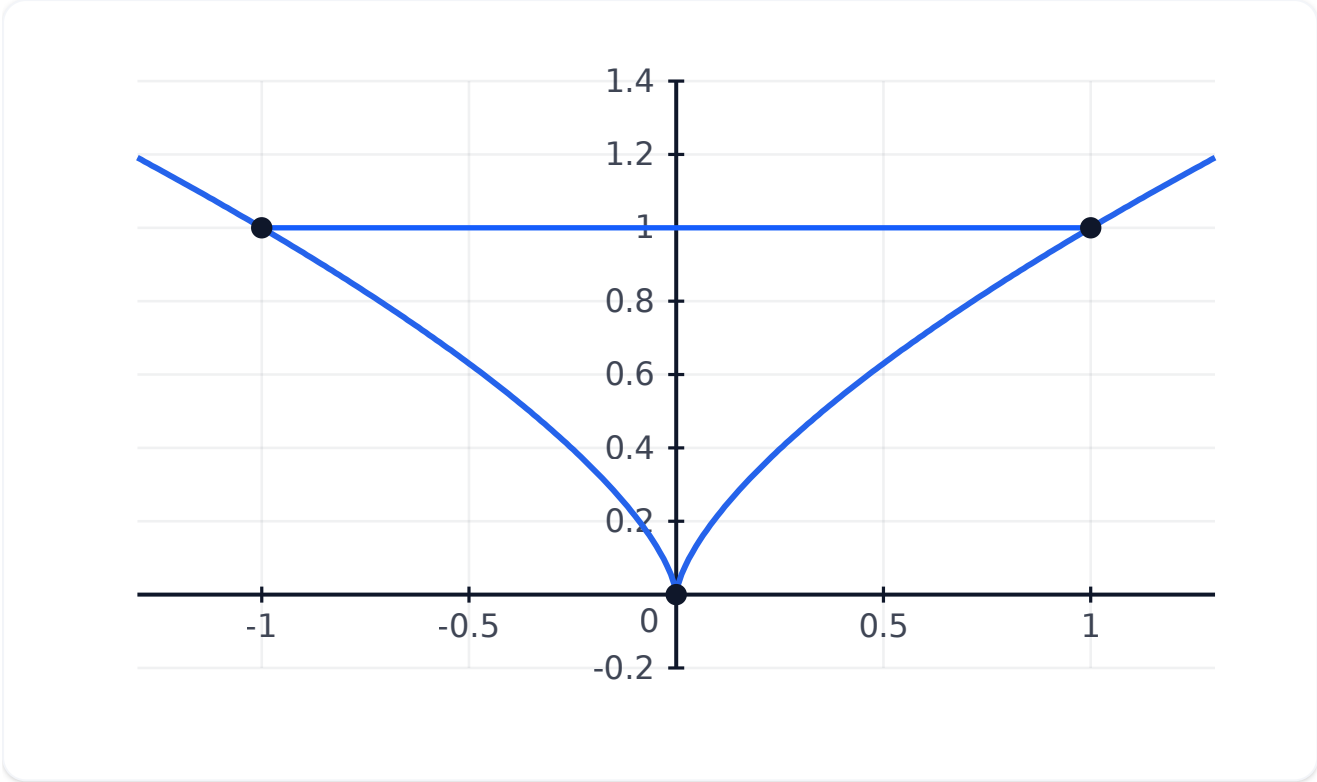


Figure 1.3.3: Failure of the Mean Value Theorem when differentiability fails at an interior point: $f(x) = x^{2/3}$ has chord slope 0 on $[-1, 1]$, but $f'(x) \neq 0$ for all $x \neq 0$.

Typical Errors

1. **Unchecked L'Hôpital Application:** Differentiating numerator and denominator when the quotient is determinate. For example, $\lim_{x \to 0} \frac{x+2}{x+1} = 2$, but applying L'Hôpital yields $\frac{1}{0}$, which is completely erroneous.
2. **Confusing Continuity with Differentiability:** Assuming that because f has no holes or jumps, its derivative must exist. The Weierstrass function is continuous everywhere and differentiable nowhere; $f(x) = |x|$ is non-differentiable at $x = 0$.
3. **Assuming f' is Continuous:** A function can be differentiable everywhere while its derivative is discontinuous. The classic counterexample $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$ has $f'(0) = 0$, but $\lim_{x \to 0} f'(x)$ does not exist due to high-frequency oscillation.
4. **Locating c at the Boundary:** Asserting that $c = a$ or $c = b$ satisfies Rolle's theorem or MVT. The theorems strictly state $c \in (a, b)$ in the **open** interval.
5. **Treating $\frac{df}{dx}$ as an Algebraic Fraction:** While Leibniz notation is suggestive, manipulating differentials algebraically without justifying the chain rule or inverse function theorem leads to severe errors in multivariable or implicit settings.

References & Further Reading

- Spivak, M. (2008). *Calculus* (4th ed.). Publish or Perish.
- Abbott, S. (2015). *Understanding Analysis* (2nd ed.). Springer.
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Integration

SECTION DETAIL

Integration: Rigorous Riemann Integral

Integration is traditionally introduced as finding the “area under a curve.” Historically, Archimedes used the method of exhaustion, but a modern mathematical treatment requires the formal definition of partitions and Riemann (or Darboux) sums to define integrability.

Partitions and Darboux Sums

Let $[a, b]$ be a closed and bounded interval.

DEFINITION 2.1.1 **Partition and Darboux Sums**

A **partition** P of $[a, b]$ is a finite set of points $\{x_0, x_1, \ldots, x_n\}$ such that:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

For each subinterval $[x_{i-1}, x_i]$, we define:

$$\Delta x_i = x_i - x_{i-1}.$$

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. For each subinterval, let:

$$m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\} \quad \text{and} \quad M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}.$$

We define the **Lower Darboux Sum** $L(f, P)$ and **Upper Darboux Sum** $U(f, P)$ as:

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i \quad \text{and} \quad U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i.$$

Pause & Predict: Partition Refinement Behavior. *Suppose you take a partition P of $[a, b]$ and insert exactly one new division point x^*, forming refinement $P^* = P \cup \{x^*\}$.*

- *Does the lower sum $L(f, P^*)$ increase, decrease, or stay the same relative to $L(f, P)$?*
- *Does the Darboux gap $U(f, P^*) - L(f, P^*)$ expand or contract? Commit your reasoning before opening the analysis below.*

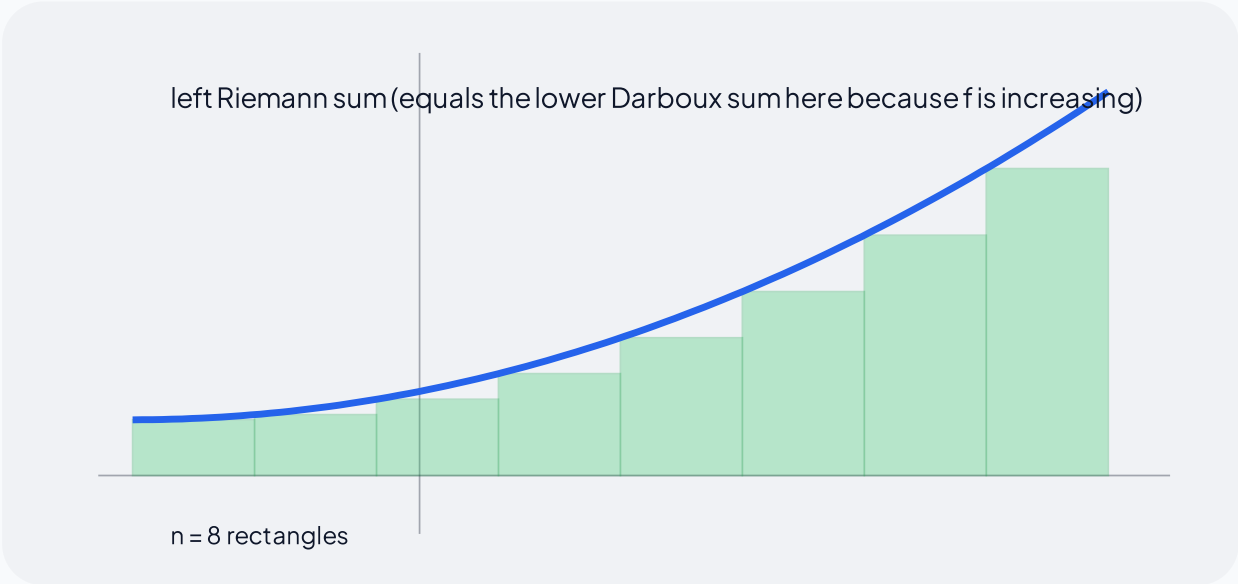
Consider the infimum of f over a subset versus the infimum over the whole interval: does $\inf_{S'} f$ increase or decrease when $S' \subseteq S$?

Analysis: Suppose $x^* \in (x_{k-1}, x_k)$. On the original interval $[x_{k-1}, x_k]$, the infimum is m_k. On the subintervals $[x_{k-1}, x^*]$ and $[x^*, x_k]$, the infima $m_k' = \inf_{x \in [x_{k-1}, x^*]} f(x)$ and $m_k'' = \inf_{x \in [x^*, x_k]} f(x)$ must satisfy $m_k' \geq m_k$ and $m_k'' \geq m_k$ because the infimum over a subset cannot decrease. Since $(x^* - x_{k-1}) + (x_k - x^*) = \Delta x_k$, we have:

$$m_k'(x^* - x_{k-1}) + m_k''(x_k - x^*) \geq m_k \Delta x_k.$$

Thus, adding points monotonically increases (or maintains) the lower sum ($L(f, P) \leq L(f, P^*)$) and monotonically decreases (or maintains) the upper sum ($U(f, P^*) \leq U(f, P)$). Refinements strictly squeeze the upper and lower Darboux bounds closer together.

REFINE A RIEMANN SUM



A finite rectangular sum approximates the area under a curve.

The lower and upper sums trap the limiting area from below and above:

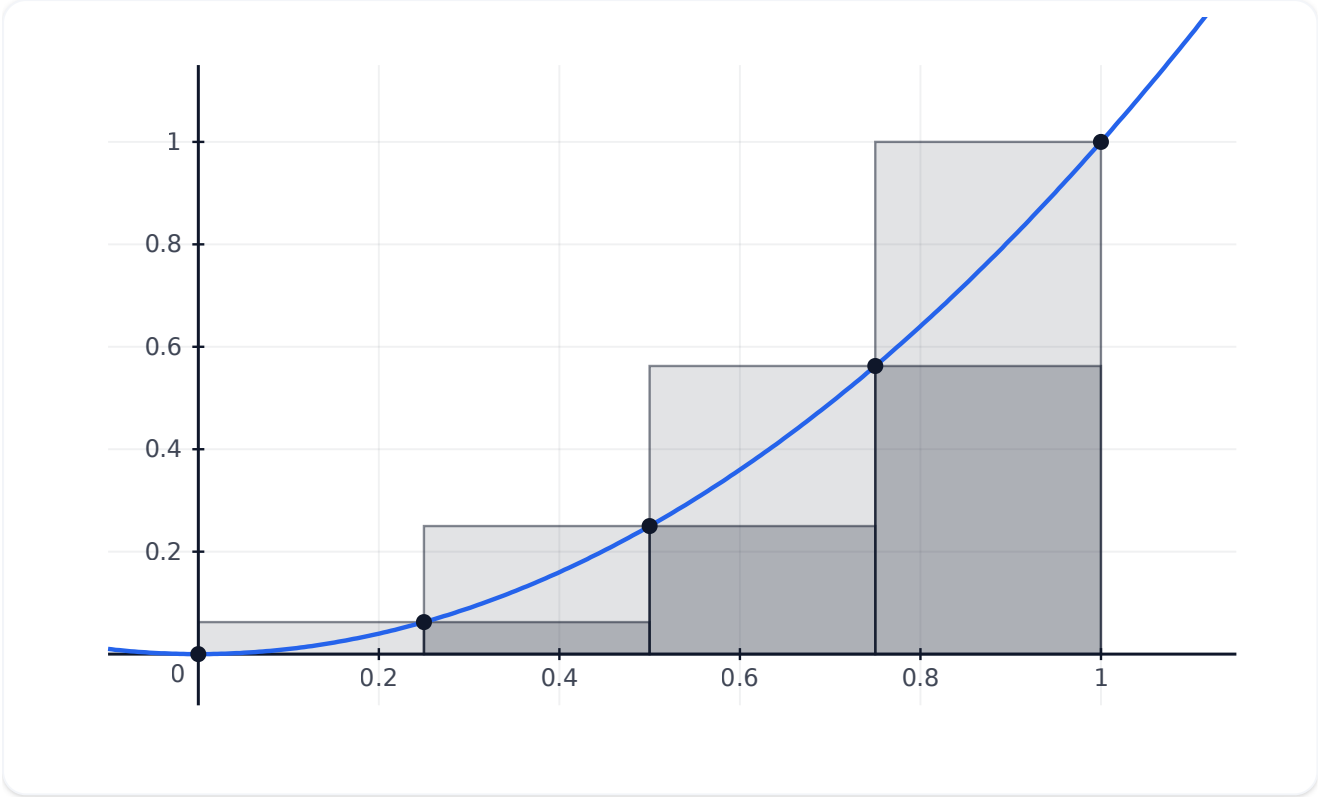


Figure 2.1.1: Darboux sums for $f(x) = x^2$ on $[0, 1]$ with $n = 4$: lower sum $L = \frac{7}{32} = 0.21875$ and upper sum $U = \frac{15}{32} = 0.46875$ sandwich $\int_0^1 x^2 \, dx = \frac{1}{3}$.

Riemann Integrability

We define the actual integral by taking supremum and infimum over all possible partitions.

DEFINITION 2.1.2 **Riemann Integrability**

Let $\mathcal{P}$ be the set of all possible partitions of $[a, b]$. We define the **lower integral** and **upper integral** of f on $[a, b]$ by:

$$\int_a^b f(x)\,dx = \sup\{L(f, P) \mid P \in \mathcal{P}\},$$

$$\int_a^b f(x)\,dx = \inf\{U(f, P) \mid P \in \mathcal{P}\}.$$

If the lower and upper integrals are equal, we say that f is **Riemann integrable** on $[a, b]$. The common value is denoted by:

$$\int_a^b f(x)\,dx.$$

Let us write a Python script to compute the upper and lower Darboux sums for $f(x) = x^2$ on $[0, 2]$ using uniform partitions of increasing size. This shows how they converge to the true value of $8/3 \approx 2.6667$.

INTERACTIVE LAB

```
import numpy as np

def f(x): return x**2

a, b = 0.0, 2.0
n_values = [4, 10, 100, 1000]

print(f"{'n':<6} | {'Lower Sum L(f, P)':<20} | {'Upper Sum U(f, P)':<20}")
print("-" * 52)

for n in n_values:
  x_coords = np.linspace(a, b, n + 1)
  lower_sum = 0.0
  upper_sum = 0.0

  for i in range(n):
      x_left = x_coords[i]
      x_right = x_coords[i+1]
      dx = x_right - x_left

      # Since f(x)=x^2 is strictly increasing on [0, 2]
      # inf is at the left endpoint, sup is at the right endpoint
      mi = f(x_left)
      Mi = f(x_right)

      lower_sum += mi * dx
      upper_sum += Mi * dx

  print(f"{n:<6} | {lower_sum:<20.6f} | {upper_sum:<20.6f}")
```

Worked Trace: Exact Analytical Evaluation of $\int_0^b x^2\,dx$ via Darboux Sums

To see the Darboux machinery in action without relying on the Fundamental Theorem of Calculus, consider $f(x) = x^2$ on $[0, b]$ partitioned into n equal subintervals of width $\Delta x = \frac{b}{n}$, with division points $x_i = \frac{ib}{n}$ for $i = 0, 1, \ldots, n$.

1. **Subinterval Bounds**: Because $f(x) = x^2$ is strictly increasing on $[0, b]$, on each interval $[x_{i-1}, x_i]$:
   - $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) = f(x_{i-1}) = \left(\frac{(i-1)b}{n}\right)^2$
   - $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) = f(x_i) = \left(\frac{ib}{n}\right)^2$

2. **Evaluating Upper and Lower Sums**: Using the summation identity $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$:

$$U(f, P_n) = \sum_{i=1}^{n} M_i \Delta x = \sum_{i=1}^{n} \left(\frac{ib}{n}\right)^2 \frac{b}{n} = \frac{b^3}{n^3} \frac{n(n+1)(2n+1)}{6} = \frac{b^3}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

Similarly, for the lower sum:

$$L(f, P_n) = \sum_{i=1}^{n} m_i \Delta x = \sum_{i=1}^{n} \left(\frac{(i-1)b}{n}\right)^2 \frac{b}{n} = \frac{b^3}{n^3} \frac{(n-1)n(2n-1)}{6} = \frac{b^3}{6}\left(2 - \frac{3}{n} +$$

3. **Bounding the Darboux Gap**:

$$U(f, P_n) - L(f, P_n) = \frac{b^3}{6}\left(\frac{6}{n}\right) = \frac{b^3}{n}$$

For any $\epsilon > 0$, choosing $n > \frac{b^3}{\epsilon}$ guarantees $U(f, P_n) - L(f, P_n) < \epsilon$.

4. **Convergence to the Riemann Integral**: Taking $n \to \infty$:

$$\sup_n L(f, P_n) = \frac{b^3}{3} \leq \int_0^b x^2 \, dx \leq \inf_n U(f, P_n) = \frac{b^3}{3}$$

Since the upper and lower integrals coincide at $\frac{b^3}{3}$, $f(x) = x^2$ is rigorously Riemann integrable with $\int_0^b x^2 \, dx = \frac{b^3}{3}$.

Counterexample & Failure Mode: Pathological Non-Integrability (Dirichlet's Function)

Does every bounded function have a Riemann integral? Consider Dirichlet's function on $[0, 1]$:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

1. **Failure Mode Analysis**:

- Between any two real numbers $x_{i-1} < x_i$, there exists an irrational number (since $\mathbb{R} \setminus \mathbb{Q}$ is dense). Thus $m_i = \inf_{[x_{i-1}, x_i]} f(x) = 0$.
- Between any two real numbers $x_{i-1} < x_i$, there also exists a rational number (since $\mathbb{Q}$ is dense). Thus $M_i = \sup_{[x_{i-1}, x_i]} f(x) = 1$.

2. **Darboux Collapse**: For **every** partition P of $[0,1]$, regardless of refinement:

$$L(f,P) = \sum_{i=1}^{n} 0 \cdot \Delta x_i = 0 \implies \int_0^1 f(x)\,dx = \sup_P L(f,P) = 0$$

$$U(f,P) = \sum_{i=1}^{n} 1 \cdot \Delta x_i = \sum_{i=1}^{n} \Delta x_i = 1 \implies \int_0^1 f(x)\,dx = \inf_P U(f,P) = 1$$

3. **Conclusion**: Because $\int_0^1 f(x)\,dx = 0 \ /= 1 = \int_0^1 f(x)\,dx$, the Darboux gap never closes ($U - L = 1$). Riemann integration completely fails for functions whose discontinuities are everywhere dense.

Integrability of Continuous Functions

Not all bounded functions are integrable (as Dirichlet's function demonstrates). However, continuity is a sufficient condition for integrability.

THEOREM 2.1.3 **Integrability of Continuous Functions**

If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then f is Riemann integrable on $[a,b]$.

Proof sketch: Since $[a,b]$ is closed and bounded, any continuous function f on it is *uniformly continuous*. This allows us to make the difference $M_i - m_i$ arbitrarily small for a sufficiently fine partition, which forces $U(f,P) - L(f,P) < \epsilon$.

REMARK

A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable if and only if its set of discontinuities has Lebesgue measure zero (Lebesgue's integrability criterion). Continuous functions and monotone functions with finitely or countably many jump discontinuities satisfy this criterion automatically, whereas Dirichlet's function fails because its discontinuity set is all of $[0,1]$ (measure 1).

Mean Value Theorem for Integrals

This theorem guarantees that a continuous function must attain its average value.

THEOREM 2.1.4 **Mean Value Theorem for Integrals**

If $f : [a, b] \to \mathbb{R}$ is continuous, then there exists some $c \in [a, b]$ such that:

$$\int_a^b f(x)\,dx = f(c)(b - a).$$

Since f is continuous on $[a, b]$, by the Extreme Value Theorem (Theorem 1.1.6), it attains a minimum value m and a maximum value M on $[a, b]$. Thus, for all $x \in [a, b]$:

$$m \leq f(x) \leq M.$$

Using the monotonicity property of the Riemann integral:

$$\int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx \implies m(b - a) \leq \int_a^b f(x)\,dx \leq M(b - a).$$

Dividing by the interval length $(b - a) > 0$:

$$m \leq \frac{1}{b - a}\int_a^b f(x)\,dx \leq M.$$

Let $k = \frac{1}{b-a}\int_a^b f(x)\,dx$. Since k is a number between the minimum m and maximum M of the continuous function f, by the Intermediate Value Theorem (Theorem 1.1.5), there must exist some $c \in [a, b]$ such that $f(c) = k$. Therefore,

$$f(c) = \frac{1}{b - a}\int_a^b f(x)\,dx \implies \int_a^b f(x)\,dx = f(c)(b - a).$$

∎

SECTION EXERCISES

Deep Practice

Compute upper and lower Darboux sums directly from partitions, bound integrals via monotonicity, and apply the Mean Value Theorem for Integrals.

Exercise 2.1.1: Darboux Sums and Exact Integrability of a Linear Function

Verification application

Let $f(x) = 2x + 1$ on $[0, 2]$, and let $P_n = \{0, 2/n, 4/n, \ldots, 2\}$ be the uniform partition into n subintervals of width $\Delta x = 2/n$. Compute $L(f, P_n)$ and $U(f, P_n)$ in closed form, verify that $U(f, P_n) - L(f, P_n) = 8/n$, and deduce $\int_0^2 (2x + 1)\, dx$ directly from Definition 2.1.2.

Since $f(x) = 2x + 1$ is strictly increasing, the infimum on $[x_{i-1}, x_i]$ occurs at x_{i-1} and the supremum occurs at x_i. Use $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

1. **Subinterval bounds:** On $[x_{i-1}, x_i]$ with $x_i = 2i/n$ and $\Delta x_i = 2/n$, f is strictly increasing, so:

$$m_i = f(x_{i-1}) = 2\left(\frac{2(i-1)}{n}\right) + 1 = 1 + \frac{4(i-1)}{n}, \qquad M_i = f(x_i) = 1 +$$

2. **Upper Darboux sum:**

$$U(f, P_n) = \sum_{i=1}^{n} \left(1 + \frac{4i}{n}\right) \frac{2}{n} = \frac{2}{n} \sum_{i=1}^{n} 1 + \frac{8}{n^2} \sum_{i=1}^{n} i = 2 + \frac{8}{n^2} \cdot \frac{n(n+1)}{2} =$$

3. **Lower Darboux sum:**

$$L(f, P_n) = \sum_{i=1}^{n} \left(1 + \frac{4(i-1)}{n}\right) \frac{2}{n} = 2 + \frac{8}{n^2} \cdot \frac{(n-1)n}{2} = 6 - \frac{4}{n}.$$

4. **Darboux gap and integral:**

$$U(f, P_n) - L(f, P_n) = \left(6 + \frac{4}{n}\right) - \left(6 - \frac{4}{n}\right) = \frac{8}{n} \to 0 \quad \text{as } n \to \infty.$$

Since $L(f, P_n) \le \int_0^2 f \le \int_0^2 f \le U(f, P_n)$ for every n, squeezing as $n \to \infty$ gives $\int_0^2 (2x + 1)\, dx = 6$.

Verification criteria. Requires explicit expressions for m_i, M_i, closed-form evaluation of both sums, verification that $U - L = 8/n$, and conclusion $\int_0^2 (2x + 1)\, dx = 6$.

Exercise 2.1.2: Mean Value Point for a Quadratic Integral

Construction application

Using the Darboux integral result $\int_0^b x^2\,dx = \frac{b^3}{3}$ established above, find the average value of $f(x) = x^2$ on $[0, 3]$ and determine every point $c \in [0, 3]$ guaranteed by Theorem 2.1.4.

Use $\int_0^3 x^2\,dx = \frac{3^3}{3} = 9$ to compute the average value $\frac{1}{3-0}\int_0^3 x^2\,dx$, then solve $c^2 = f_{\text{avg}}$ on $[0, 3]$.

1. **Compute the integral from the Darboux result:** Substituting $b = 3$ into $\int_0^b x^2\,dx = \frac{b^3}{3}$ yields:

$$\int_0^3 x^2\,dx = \frac{3^3}{3} = 9.$$

2. **Average value:**

$$f_{\text{avg}} = \frac{1}{3-0}\int_0^3 x^2\,dx = \frac{9}{3} = 3.$$

3. **Solve $f(c) = 3$:**

$$c^2 = 3 \implies c = \sqrt{3} \text{ or } c = -\sqrt{3}.$$

4. **Filter to $[0, 3]$:** Only $c = \sqrt{3} \approx 1.732$ lies in $[0, 3]$.

Verification criteria. Requires finding $\int_0^3 x^2\,dx = 9$ and $f_{\text{avg}} = 3$, solving $c^2 = 3$, and selecting the root $c = \sqrt{3} \in [0, 3]$.

Exercise 2.1.3: Telescoping Darboux Gap for Monotone Functions

Proof synthesis

Let $f : [a, b] \to \mathbb{R}$ be monotonically increasing on $[a, b]$, and let P_n be the uniform partition of $[a, b]$ into n subintervals of width $\Delta x = (b - a)/n$. Prove that

$$U(f, P_n) - L(f, P_n) = \frac{b - a}{n}\big(f(b) - f(a)\big),$$

and deduce that every monotone function on $[a, b]$ is Riemann integrable—even if it has jump discontinuities.

On each subinterval $[x_{i-1}, x_i]$, monotonicity forces $m_i = f(x_{i-1})$ and $M_i = f(x_i)$. Factor out Δx and telescope the sum.

1. **Subinterval extrema:** Because f is monotonically increasing on $[a, b]$, for every subinterval $[x_{i-1}, x_i]$ we have $f(x_{i-1}) \leq f(x) \leq f(x_i)$, so $m_i = f(x_{i-1})$ and $M_i = f(x_i)$.
2. **Subtract the Darboux sums:** Since $\Delta x_i = \frac{b-a}{n}$ for all i:

$$U(f, P_n) - L(f, P_n) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i = \frac{b-a}{n} \sum_{i=1}^{n} \big(f(x_i) - f(x_{i-1})\big).$$

3. **Telescope the sum:**

$$\sum_{i=1}^{n} \big(f(x_i) - f(x_{i-1})\big) = \big(f(x_1) - f(x_0)\big) + \big(f(x_2) - f(x_1)\big) + \cdots + \big(f(x$$

   Hence $U(f, P_n) - L(f, P_n) = \frac{b-a}{n}\big(f(b) - f(a)\big)$.
4. **Oscillation criterion:** Given $\epsilon > 0$, choose $n > \frac{(b-a)(f(b)-f(a))}{\epsilon}$. Then $U(f, P_n) - L(f, P_n) < \epsilon$, so f is Riemann integrable by Definition 2.1.2. Continuity was never used!

Verification criteria. Requires identifying $m_i = f(x_{i-1})$ and $M_i = f(x_i)$ by monotonicity, telescoping the sum to $f(b) - f(a)$, and choosing n so that $U - L < \epsilon$.

Exercise 2.1.4: Diagnosing Non-Integrability of Nowhere-Continuous Functions

Description transfer

Examine the Darboux upper and lower integrals of Dirichlet's indicator function on $[0, 1]$, and complete the midpoint Riemann sum implementation below.

KNOWLEDGE CHECK

Let $f(x) = 1$ for rational x and $f(x) = 0$ for irrational x on $[0, 1]$. Evaluate each statement regarding Darboux integrals and integrability:

Answer: The lower Darboux integral equals 0 because the infimum on every non-empty subinterval is 0.; The upper Darboux integral equals 1 because the supremum on every non-empty subinterval is 1.

Every subinterval contains both rational and irrational numbers, so $m_i = 0$ and $M_i = 1$. Thus $L(f, P) = 0$ and $U(f, P) = 1$ for every partition P; the lower integral is 0, the upper integral is 1, and f is not Riemann integrable.

COMPLETE THE MIDPOINT RIEMANN SUM

```
def midpoint_riemann_sum(f, a, b, n):
  dx = (b - a) / n
  total_area = 0.0
  for i in range(n):
      # Calculate midpoint of subinterval i
      mid = a + (i + 0.5) * dx
      total_area += f(mid) * dx
  return total_area
```

The midpoint of subinterval i (from i*dx to (i+1)*dx) is (i + 0.5)*dx.

References & Further Reading

- Spivak, M. (2008). *Calculus* (4th ed.), Chapter 13 ("Integrals"). Publish or Perish.
- Apostol, T. M. (1967). *Calculus, Volume 1: One-Variable Calculus, with an Introduction to Linear Algebra* (2nd ed.), Chapter 1 & Chapter 2 ("The Concepts of Integral Calculus" and "Some Applications of Integration"). Wiley.
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SECTION DETAIL

The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus (FTC) is the central theorem of calculus, establishing the inverse relationship between differentiation (slopes) and integration (accumulation).

FTC Part 1: Derivative of the Accumulation Function

This part of the theorem states that the rate of change of the area under a curve is equal to the height of the curve.

THEOREM 2.2.1 **Fundamental Theorem of Calculus, Part 1**

Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function, and let f be continuous on $[a, b]$. Define the accumulation function $F : [a, b] \to \mathbb{R}$ by:

$$F(x) = \int_a^x f(t)\,dt.$$

Then F is continuous on $[a, b]$, differentiable on (a, b), and:

$$F'(x) = f(x) \quad \text{for all } x \in (a, b).$$

We prove $F'(x) = f(x)$ at any $x \in (a, b)$. Using the limit definition of the derivative:

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}.$$

Using the additivity property of the Riemann integral, $F(x+h) - F(x) = \int_x^{x+h} f(t)\, dt$. Thus:

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \int_x^{x+n} f(t)\, dt.$$

Since f is continuous on the interval $[x, x+h]$ (or $[x+h, x]$ if $h < 0$), by the Mean Value Theorem for Integrals (Theorem 2.1.4), there exists some c_h between x and $x+h$ such that:

$$\int_x^{x+n} f(t)\, dt = f(c_h) \cdot h \implies \frac{1}{h} \int_x^{x+n} f(t)\, dt = f(c_h).$$

Now take the limit as $h \to 0$. Since c_h is trapped between x and $x+h$, by the Squeeze Theorem (Theorem 1.1.3), $\lim_{h \to 0} c_h = x$. Since f is continuous, we have:

$$F'(x) = \lim_{h \to 0} f(c_h) = f\left(\lim_{h \to 0} c_h\right) = f(x).$$

This completes the proof.

∎

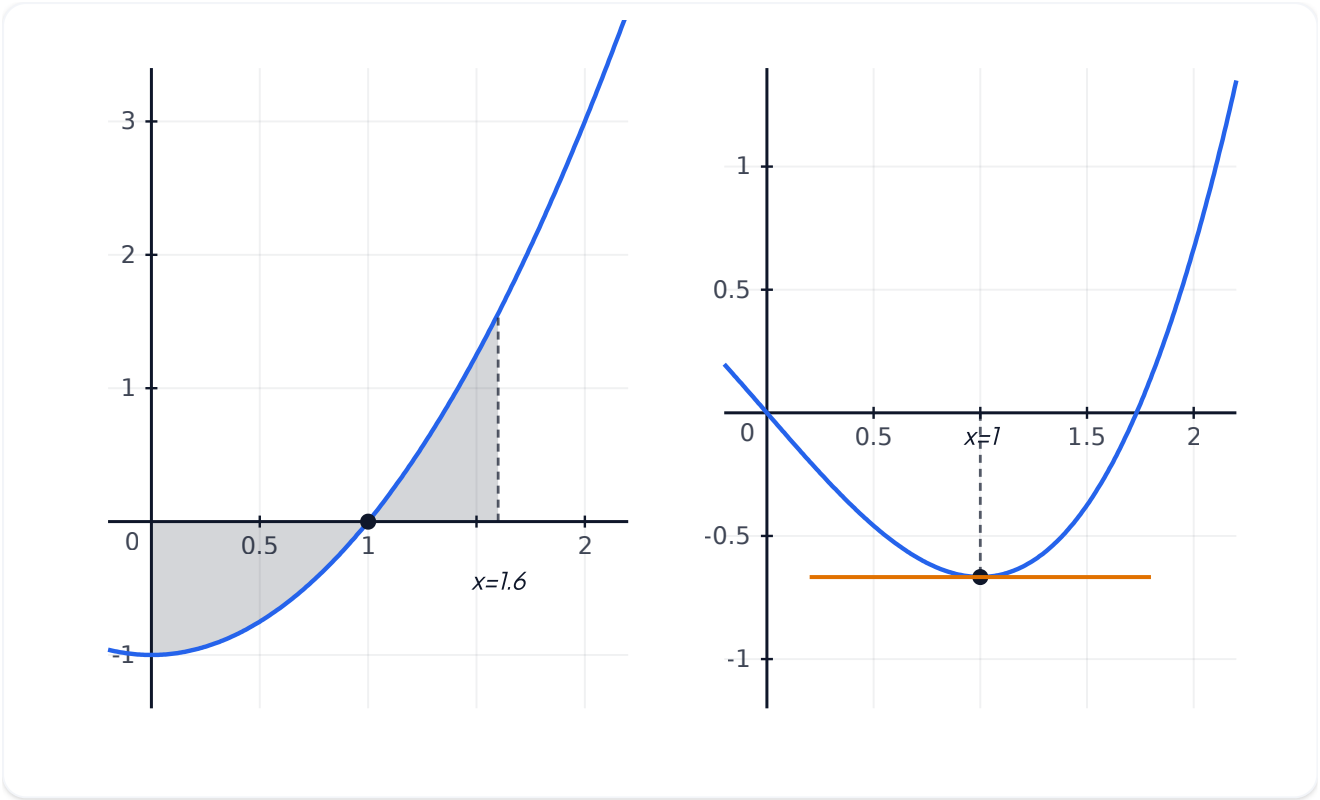


Figure 2.2.1: The Fundamental Theorem of Calculus: the accumulation function $A(x) = \int_0^x (t^2 - 1)\,dt$ reaches its global minimum at $x = 1$, where the rate of change $f(1) = 0$.

Counterexample: Discontinuity Destroys Differentiability

The continuity of f in FTC Part 1 is strictly necessary, not an optional convenience.

> ***Counterexample.*** *Consider the signum step function on* $[-1, 1]$:
>
> $$f(t) = \begin{cases} -1 & t < 0 \\ 1 & t \geq 0 \end{cases}$$
>
> *Its accumulation function* $F(x) = \int_1^x f(t)\,dt = |x| - 1$ *is continuous everywhere, but has a sharp corner at* $x = 0$ *where* $F'(0)$ *does not exist. Thus* $F'(0) \not= f(0)$. *Differentiability of the accumulation function requires* f *to be continuous.*

REMARK

In the Riemann setting, FTC Part 1 requires f to be continuous at x so that the average value on $[x, x+h]$ converges to $f(x)$; in Lebesgue theory, an absolutely continuous function F is differentiable almost everywhere with $\int_a^b F'(t)\,dt = F(b) - F(a)$.

Worked trace: variable limits of integration with the Chain Rule. Compute the derivative of $G(x) = \int_1^{x^3} \sqrt{1+t^4}\,dt$.

1. Let $u = g(x) = x^3$ and define $F(u) = \int_1^u \sqrt{1+t^4}\,dt$.
2. By FTC Part 1, $\frac{dF}{du} = \sqrt{1+u^4}$.
3. By the Chain Rule, $\frac{dG}{dx} = \frac{dF}{du} \cdot \frac{du}{dx} = \sqrt{1+u^4} \cdot (3x^2)$.
4. Substitute $u = x^3$:

$$G'(x) = 3x^2\sqrt{1+(x^3)^4} = 3x^2\sqrt{1+x^{12}}.$$

Prediction — answer before expanding. What happens if we apply FTC Part 2 blindly to $\int_{-1}^{1} \frac{1}{x^2}\,dx$ using the antiderivative $-1/x$? What is the fatal flaw in the calculation?

A naive substitution gives $\left[-\frac{1}{x}\right]_{-1}^{1} = (-1) - (1) = -2$. However, the integrand $1/x^2 > 0$ everywhere, so its integral cannot be negative! The fatal flaw: FTC Part 2 strictly requires f to be continuous on the entire interval $[a, b]$. Because $1/x^2$ has an infinite discontinuity at $x = 0$, it is not Riemann integrable on $[-1, 1]$ without improper integral regularization (which diverges to $+\infty$).

FTC Part 2: Evaluation of Definite Integrals

This part provides the algebraic shortcut to evaluate definite integrals using antiderivatives.

THEOREM 2.2.2 **Fundamental Theorem of Calculus, Part 2**

Builds on: Theorem 2.2.1

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. If G is any antiderivative of f on $[a, b]$ (that is, $G'(x) = f(x)$ for all $x \in [a, b]$), then:

$$\int_a^b f(x)\,dx = G(b) - G(a).$$

Let $F(x) = \int_a^x f(t)\,dt$. By Theorem 2.2.1, $F'(x) = f(x)$. Since G is also an antiderivative of f, we have $G'(x) = f(x) = F'(x)$ for all $x \in [a, b]$. Therefore, $(F - G)'(x) = 0$ for all $x \in [a, b]$. By Corollary 1.3.3, there exists a constant C such that:

$$F(x) = G(x) + C \quad \text{for all } x \in [a, b].$$

To find C, we evaluate at $x = a$:

$$F(a) = G(a) + C \implies \int_a^a f(t)\,dt = G(a) + C \implies 0 = G(a) + C \implies C = -G(a)$$

Substituting C back into the equation:

$$F(x) = G(x) - G(a).$$

Now, evaluating at $x = b$:

$$F(b) = G(b) - G(a) \implies \int_a^b f(t)\,dt = G(b) - G(a).$$

This completes the proof.

∎

Integration Techniques from Derivative Rules

By reversing derivative rules, we obtain powerful integration tools.

Substitution Rule (Reverse Chain Rule)

If $u = g(x)$ is differentiable and f is continuous, then:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Integration by Parts (Reverse Product Rule)

If $u(x)$ and $v(x)$ are differentiable, then:

$$\int u\,dv = uv - \int v\,du.$$

Let us use SymPy to find the antiderivative of $x\ln(x)$ using integration by parts, and evaluate its definite integral from 1 to e.

INTERACTIVE LAB

```
import sympy as sp

x = sp.Symbol('x')
f = x * sp.log(x)

# Antiderivative (Indefinite integral)
F = sp.integrate(f, x)
print(f"Indefinite integral of x*ln(x): {F}")

# Definite integral from 1 to e
a, b = 1, sp.E
definite_integral = sp.integrate(f, (x, a, b))
print(f"Definite integral from 1 to e: {definite_integral}")
print(f"Numerical value: {float(definite_integral):.6f}")
```

SECTION EXERCISES

Deep Practice

Apply both parts of the Fundamental Theorem of Calculus, differentiate variable-limit accumulation functions, and evaluate definite integrals by substitution and integration by parts.

Exercise 2.2.1: Differentiating an Accumulation Function with Two Variable Limits

Construction application

Differentiate $H(x) = \int_{x}^{\sin x^2} \cos(t^2)\,dt$ with respect to x, justifying each step via Theorem 2.2.1 and Theorem 1.2.4.

Split the integral at a constant base point 0: $H(x) = \int_0^{x^2} \cos(t^2)\,dt - \int_0^{\sin x} \cos(t^2)\,dt$, then apply FTC Part 1 and the Chain Rule to each term.

1. **Split at a fixed base point:** Since $f(t) = \cos(t^2)$ is continuous on $\mathbb{R}$, define $F(u) = \int_0^u \cos(t^2)\,dt$. By Theorem 2.2.1, $F'(u) = \cos(u^2)$.
2. **Express H in terms of F:**

$$H(x) = \int_0^{x^2} \cos(t^2)\,dt - \int_0^{\sin x} \cos(t^2)\,dt = F(x^2) - F(\sin x).$$

3. **Differentiate via Theorem 1.2.4:**

$$H'(x) = F'(x^2)\cdot(2x) - F'(\sin x)\cdot(\cos x) = 2x\cos(x^4) - \cos(\sin^2 x)\cos$$

Verification criteria. Requires decomposing into two fixed-lower-limit integrals, applying Theorem 2.2.1, and multiplying each term by the derivative of its limit of integration.

Exercise 2.2.2: Definite Substitution with Transformed Limits

Verification application

Evaluate $\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx$ by substitution, transforming the limits of integration explicitly and invoking Theorem 2.2.2.

Set $u = x^2$ so $du = 2x\,dx$, and map the endpoints $x = 0 \mapsto u = 0$ and $x = \sqrt{\pi} \mapsto u = \pi$.

1. **Substitute** $u = x^2$: Notice $du = 2x\,dx \implies x\,dx = \frac{1}{2}du$.
2. **Transform endpoints:** At $x = 0, u = 0^2 = 0$; at $x = \sqrt{\pi}, u = (\sqrt{\pi})^2 = \pi$.
3. **Evaluate the transformed definite integral by** Theorem 2.2.2**:**

$$\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx = \frac{1}{2}\int_0^{\pi} \sin(u)\,du = \frac{1}{2}\big[-\cos u\big]_0^{\pi} = \frac{1}{2}\big(-\cos\pi - (-\cos$$

Verification criteria. Requires explicit differential $du = 2x\,dx$, transformed bounds $[0, \pi]$, and exact evaluation to 1.

Exercise 2.2.3: Integration by Parts and Product-Rule Reversal

Construction transfer

Derive the Integration by Parts formula from the Product Rule (Theorem 1.2.3), then evaluate $\int_0^1 xe^{2x}\,dx$ in exact closed form.

COMPLETE THE INTEGRATION BY PARTS FORMULA

We know the product rule: $\frac{d}{dx}(u \cdot v) = u' \cdot v + u \cdot v'$.
Integrating both sides: $u \cdot v = \int (u' \cdot v)\,dx + \int (u \cdot v')\,dx$.
Rearranging terms gives the Integration by Parts formula:
$\int (u \cdot v')\,dx = u \cdot v - \int (u' \cdot v)\,dx$

Integrating the product rule $\frac{d}{dx}(u \cdot v) = u' \cdot v + u \cdot v'$ and rearranging gives $\int (u \cdot v')\,dx = u \cdot v - \int (u' \cdot v)\,dx$.

Choose $u = x$ and $dv = e^{2x}\,dx$, so $du = dx$ and $v = \frac{1}{2}e^{2x}$.

1. **Set parts:** Let $u = x$ $(du = dx)$ and $dv = e^{2x}\,dx$ $(v = \frac{1}{2}e^{2x})$.
2. **Apply Integration by Parts on** $[0, 1]$:

$$\int_0^1 xe^{2x}\,dx = \left[\frac{1}{2}xe^{2x}\right]_0^1 - \int_0^1 \frac{1}{2}e^{2x}\,dx = \frac{1}{2}e^2 - \left[\frac{1}{4}e^{2x}\right]_0^1 = \frac{1}{2}e^2 - \left(\frac{1}{4}e\right.$$

Verification criteria. Requires identifying u, dv, du, v, applying the boundary term $[uv]_0^1$, and simplifying to $(e^2 + 1)/4$.

Exercise 2.2.4: Variable-Limit Differentiation Check

Verification application

Verify the Leibniz-Chain Rule decomposition for a polynomial accumulation function with two variable limits:

KNOWLEDGE CHECK

Let $H(x) = \int_{x^2}^{x^3} t^3\, dt$. Evaluate each statement regarding its derivative $H'(x)$ and method of computation:

Answer: Applying the Fundamental Theorem of Calculus with the Chain Rule yields $H'(x) = 3x^{11} - 2x^7$.; By decomposing $H(x) = F(x^3) - F(x^2)$, differentiation produces $(x^3)^3 \cdot (3x^2) - (x^2)^3 \cdot (2x)$.

Write $H(x) = F(x^3) - F(x^2)$ with $F'(u) = u^3$. The chain rule gives $H'(x) = (x^3)^3(3x^2) - (x^2)^3(2x) = 3x^{11} - 2x^7$.

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SECTION DETAIL

Applications of Integration

Where the previous lessons established the existence and evaluation of the definite integral, this lesson turns to geometric and physical modeling. The central modeling decision is to choose a thin slice or element, express its measure in terms of the integration variable, and sum the elements in a limit.

Area Between Curves

If f and g are continuous on $[a, b]$ and $f(x) \geq g(x)$ there, the region between their graphs has area

$$A = \int_a^b \big(f(x) - g(x)\big)\, dx.$$

The order is not optional: if the graphs cross, split the interval at every intersection or integrate $|f - g|$. When horizontal slices are simpler, solve for x as a function of y and use $A = \int_c^d (x_{\text{right}}(y) - x_{\text{left}}(y))\, dy$.

The shaded region is the geometric object represented by the integrand:

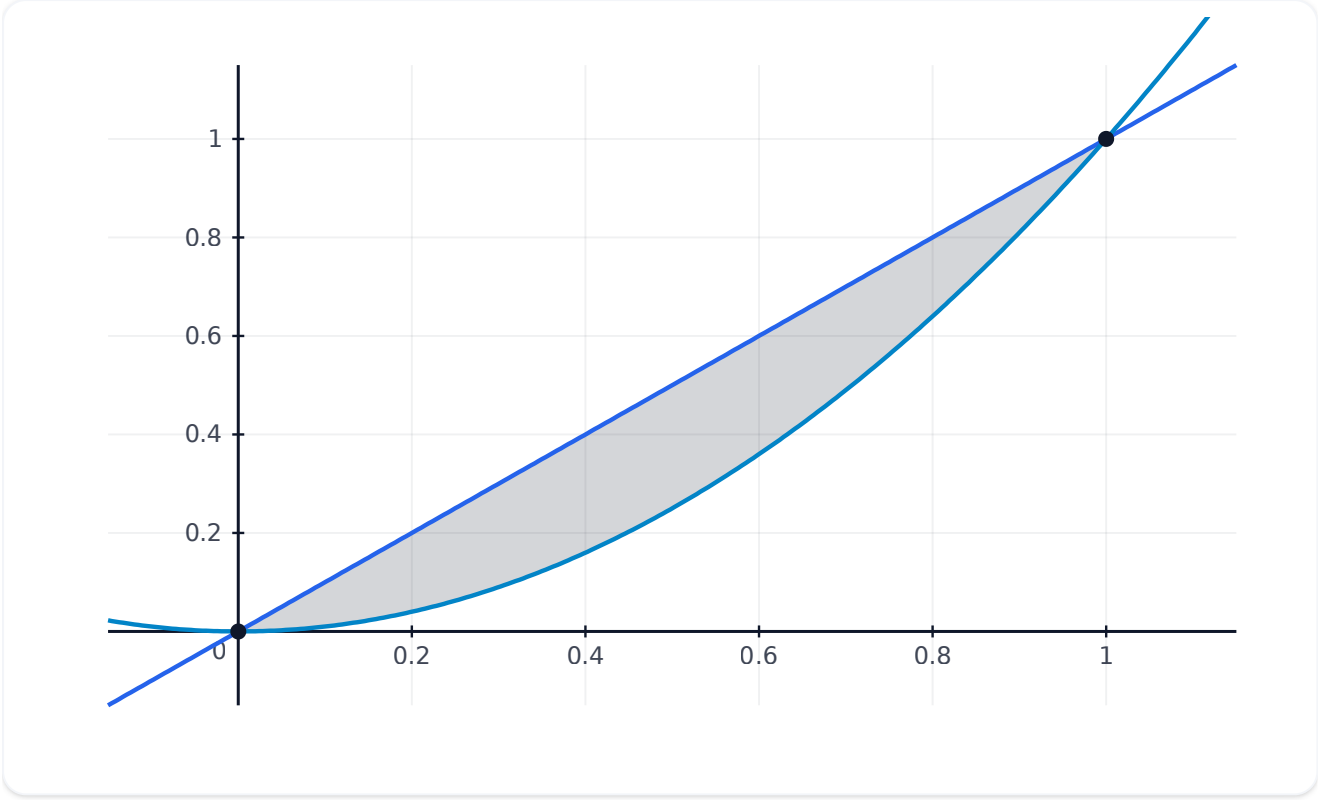


Figure 2.3.1: Area bounded between curves: the region enclosed by $y = x$ and $y = x^2$ on $[0, 1]$ has area $\int_0^1 (x - x^2)\,dx = \frac{1}{6}$.

For $y = x$ and $y = x^2$, the intersections are $x = 0, 1$, and the line is above the parabola on that interval. Hence

$$A = \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}.$$

> ***Pause & Predict: Crossing Curves & The Vanishing Net Area Trap.*** *Consider computing the geometric area between $f(x) = \sin(x)$ and $g(x) = \cos(x)$ on $[0, 2\pi]$. What happens if you directly evaluate $\int_0^{2\pi} (\sin x - \cos x)\,dx$ without finding intersection points? Commit your reasoning before opening the analysis below.*

Recall that geometric area is strictly positive. If the curves cross, what does naive signed integration do to the opposing regions?

Analysis: Directly integrating gives:

$$\int_0^{2\pi} (\sin x - \cos x)\, dx = [-\cos x - \sin x]_0^{2\pi} = (-1 - 0) - (-1 - 0) = 0!$$

The naive algebraic integral yields **zero** because the positive region between $x = \pi/4$ and $x = 5\pi/4$ is exactly cancelled by the opposite regions on $[0, \pi/4]$ and $[5\pi/4, 2\pi]$. Geometric area is *strictly positive*: one must split at the intersection points $x = \pi/4, 5\pi/4$ to integrate $|f(x) - g(x)|$:

$$A = \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx + \int_{5\pi/4}^{2\pi} (\cos x - \sin x)\, dx =$$

CURVE CROSSING CHECK

Consider $f(x) = x^3$ and $g(x) = x$ on $[-1, 1]$. Evaluate each statement regarding the enclosed geometric area and integration:

Answer: The true geometric enclosed area is 0.5, computed by integrating $|x - x^3|$ across $[-1, 1]$.; By symmetry between $[-1, 0]$ and $[0, 1]$, the total area equals $2\int_0^1 (x - x^3)\, dx$.

By symmetry around x=0, f(x)-g(x) is an odd function so the raw integral is 0. But the geometric area is $2\int_0^1 (x - x^3)\, dx = 2[x^2/2 - x^4/4]_0^1 = 2(1/2 - 1/4) = 0.5$.

Volumes by Slicing

Let a solid have cross-sectional area $A(x)$ perpendicular to the x-axis. If A is integrable, Cavalieri's slicing principle gives

THEOREM 2.3.1 **Volume by Slicing**

$$V = \int_a^b A(x)\,dx.$$

Revolving a region about the x-axis produces disks of area $\pi[R(x)]^2$ when there is no hole, and washers of area $\pi([R(x)]^2 - [r(x)]^2)$ when the inner radius is $r(x)$. Shells instead use cylindrical strips: revolving a vertical strip of radius x and height $h(x)$ about the y-axis gives

$$V = 2\pi \int_a^b x h(x)\,dx.$$

For the region under $y = 2x - x^2$ on $[0, 2]$, rotation about the x-axis gives

$$V = \pi \int_0^2 (2x - x^2)^2\,dx = \frac{64\pi}{15}.$$

Arc Length and Surface Area

Approximate a smooth graph by line segments. For $y = f(x)$, the distance element satisfies

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + [f'(x)]^2}\,dx.$$

Therefore:

DEFINITION 2.3.2 **Arc Length of a Graph**

If f is continuously differentiable on $[a, b]$, its graph has length

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx.$$

Rotating the graph about the x-axis sweeps out a band with circumference $2\pi f(x)$, so for a nonnegative f,

$$S = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\,dx.$$

These integrals frequently have no elementary antiderivative; numerical quadrature is then part of the mathematical solution, not a failure of the model.

Worked Trace: Exact Arc Length Evaluation for $y = \frac{2}{3}x^{3/2}$

Consider finding the exact arc length of the curve $y = \frac{2}{3}x^{3/2}$ on the interval $[0, 3]$.

1. **Differentiate the Curve**:

$$\frac{dy}{dx} = \frac{2}{3} \cdot \frac{3}{2}x^{1/2} = \sqrt{x}.$$

2. **Formulate the Arc Length Differential**:

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + (\sqrt{x})^2}\,dx = \sqrt{1 + x}\,dx.$$

3. **Set Up the Definite Integral**:

$$L = \int_0^3 \sqrt{1 + x}\,dx.$$

4. **Integrate via Substitution**: Let $u = 1 + x$, so $du = dx$. When $x = 0, u = 1$; when $x = 3, u = 4$:

$$L = \int_1^4 u^{1/2}\,du = \left[\frac{2}{3}u^{3/2}\right]_1^4 = \frac{2}{3}(4^{3/2} - 1^{3/2}) = \frac{2}{3}(8 - 1) = \frac{14}{3} \approx 4.6667.$$

Counterexample & Failure Mode: Gabriel's Horn (Finite Volume, Infinite Surface Area)

When integration applications are extended to unbounded domains, geometric intuition can break down dramatically. Consider revolving $f(x) = \frac{1}{x}$ on the interval $[1, \infty)$ about the x-axis.

1. **Volume via Disk Slicing**:

$$V = \pi \int_1^\infty [f(x)]^2\,dx = \pi \int_1^\infty \frac{1}{x^2}\,dx = \pi \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b = \pi \lim_{b\to\infty} \left(1 - \frac{1}{b}\right) = \pi.$$

   The volume is finite! It could be filled with exactly π cubic units of paint.

2. **Surface Area via Revolving Bands**:

$$S = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \left(-\frac{1}{x^2}\right)^2}\,dx = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx.$$

   Because $\sqrt{1 + \frac{1}{x^4}} > 1$ for all $x \geq 1$:

$$S > 2\pi \int_1^\infty \frac{1}{x}\,dx = 2\pi \lim_{b\to\infty} [\ln x]_1^b = 2\pi \lim_{b\to\infty} \ln b = \infty.$$

3. **The Painter's Paradox Breakdown**: The surface area diverges to infinity while the volume remains strictly finite (π). This paradox demonstrates that three-dimensional measure (volume $\sim \int x^{-2}$) converges much faster than two-dimensional boundary measure (surface area $\sim \int x^{-1}$), warning against conflating surface and volume scalings on unbounded regimes.

Mass, Work, and Average Value

For a one-dimensional object with linear density $\rho(x)$, a slice of width dx has mass $dm = \rho(x)\,dx$, giving

$$m = \int_a^b \rho(x)\,dx.$$

If a variable force $F(x)$ moves an object from a to b in the positive direction, the work is

$$W = \int_a^b F(x)\,dx.$$

The same accumulation logic yields the average value of a continuous function:

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Worked example. A two-metre rod has density $\rho(x) = 1 + x$ kg/m, where x is measured in metres from one end. Its mass is $\int_0^2 (1 + x)\,dx = [x + x^2/2]_0^2 = 4$ kg. Its average density is $4/2 = 2$ kg/m. Dividing by length is appropriate for average density, not for the total mass.

SECTION EXERCISES

Deep Practice

Set up and evaluate definite integrals for area between curves, volumes by disks/washers and shells, arc length, and physical accumulation.

Exercise 2.3.1: Area Between Two Intersecting Parabolas

Construction application

Find the exact area of the bounded region enclosed by the curves $y = 4 - x^2$ and $y = x^2 - 4$.

Solve $4 - x^2 = x^2 - 4$ to locate the intersection points, verify which curve lies above on that interval, and integrate the difference.

1. **Intersections:** Set $4 - x^2 = x^2 - 4 \implies 2x^2 = 8 \implies x = \pm 2$.
2. **Upper vs. lower curve:** On $[-2, 2]$, at $x = 0$ we have $4 - 0^2 = 4 > -4 = 0^2 - 4$, so $y = 4 - x^2$ is the upper curve.
3. **Evaluate the area integral (using even symmetry):**

$$A = \int_{-2}^{2} \left((4 - x^2) - (x^2 - 4) \right) dx = 2 \int_{0}^{2} (8 - 2x^2)\, dx = 2 \left[8x - \frac{2x^3}{3} \right]_{0}^{2} =$$

Verification criteria. Requires finding $x = \pm 2$, setting up $\int_{2}^{2} (8 - 2x^2)\, dx$, and evaluating to $64/3$.

Exercise 2.3.2: Volume of Revolution by Cylindrical Shells and Washers

Verification application

Consider the planar region bounded by $y = x$ and $y = x^2$ on $[0, 1]$. **(a)** Set up and evaluate the volume when revolved about the x-axis using washers (Theorem 2.3.1). **(b)** Set up and evaluate the volume when revolved about the y-axis using cylindrical shells.

MODEL SELECTION

A planar region is revolved around the y-axis with vertical strips of height $h(x)$ at distance x. Evaluate each statement:

Answer: Each thin vertical strip generates a cylindrical shell of circumference $2\pi x$, thickness dx, and height $h(x)$.; The total volume accumulated across the radial interval $[a, b]$ is given by $2\pi \int_a^b xh(x)\,dx$.

A vertical strip becomes a cylindrical shell with radius x, circumference $2\pi x$, and height $h(x)$, so its volume contribution is $2\pi x h(x)\,dx$.

SET UP THE WASHER CROSS-SECTIONAL AREA

When rotating the region between $y = x$ and $y = x^2$ on $[0, 1]$ about the x-axis, the outer radius is $R(x) = x$ and the inner radius is $r(x) = x^2$.
Thus the washer cross-sectional area is $A(x) = \pi([R(x)]^2 - [r(x)]^2)$, where the outer term $[R(x)]^2 = x^2$ and $[r(x)]^2 = x^4$.

Since $y = x$ lies above $y = x^2$ on $[0, 1]$, the squared outer radius is $[R(x)]^2 = x^2$ and the squared inner radius is $[r(x)]^2 = (x^2)^2 = x^4$.

For (a), outer radius $R(x) = x$ and inner radius $r(x) = x^2$. For (b), shell radius is x and shell height is $h(x) = x - x^2$.

- **(a) About the x-axis (washers):**

$$V_x = \pi \int_0^1 \left(x^2 - (x^2)^2\right) dx = \pi \int_0^1 (x^2 - x^4)\, dx = \pi \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{2\pi}{15}.$$

- **(b) About the y-axis (cylindrical shells):**

$$V_y = 2\pi \int_0^1 x(x - x^2)\, dx = 2\pi \int_0^1 (x^2 - x^3)\, dx = 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 2\pi \cdot$$

Verification criteria. Requires exact integrals and values $V_x = 2\pi/15$ and $V_y = \pi/6$.

Exercise 2.3.3: Exact Arc Length of a Semicubical Parabola

Construction transfer

Using Definition 2.3.2, compute the exact arc length of the curve $y = \frac{2}{3}x^{3/2}$ on the interval $[0, 3]$.

Differentiate to get $y' = x^{1/2}$, so $1 + (y')^2 = 1 + x$. Integrate $\sqrt{1 + x}$ on $[0, 3]$.

1. **Derivative and arc-length integrand:**

$$y'(x) = \frac{2}{3}\cdot\frac{3}{2}x^{1/2} = \sqrt{x} \implies 1 + \big(y'(x)\big)^2 = 1 + x.$$

2. **Integrate via Definition 2.3.2:**

$$L = \int_0^3 \sqrt{1+x}\,dx = \left[\frac{2}{3}(1+x)^{3/2}\right]_0^3 = \frac{2}{3}\left(4^{3/2} - 1^{3/2}\right) = \frac{2}{3}(8-1) = \frac{1}{3}$$

Verification criteria. Requires computing $1 + (y')^2 = 1 + x$ and evaluating the integral to $14/3$.

Exercise 2.3.4: Work Required to Pump Liquid from a Cylindrical Tank

Construction synthesis

A vertical cylindrical tank of radius $R = 2\ \text{m}$ and height $H = 5\ \text{m}$ is filled to the brim with water of weight density $\gamma = 9800\ \text{N/m}^3$. Set up and evaluate the definite integral for the total work required to pump all the water over the top rim of the tank.

Slice the water horizontally at depth $y \in [0, 5]$ below the rim. Each disk slice has volume $\pi R^2\,dy$, weight $\gamma\pi R^2\,dy$, and is lifted a distance y.

1. **Slice geometry at depth $y \in [0, 5]$ below the top rim:**
   - Cross-sectional area: $A(y) = \pi R^2 = 4\pi \ \mathrm{m}^2$.
   - Slice volume: $dV = 4\pi \, dy \ \mathrm{m}^3$.
   - Slice weight (force): $dF = \gamma \, dV = 4\pi\gamma \, dy \ \mathrm{N}$.
   - Lifting distance to the rim: $y \ \mathrm{m}$.
2. **Accumulate work:**

$$W = \int_0^5 4\pi\gamma \, y \, dy = 4\pi\gamma \left[\frac{y^2}{2}\right]_0^5 = 50\pi\gamma.$$

3. **Substitute $\gamma = 9800 \ \mathrm{N/m}^3$:**

$$W = 50\pi(9800) = 490{,}000\pi \ \mathrm{J} \approx 1.539 \times 10^6 \ \mathrm{J}.$$

Verification criteria. Requires defining the depth coordinate y, expressing slice weight $4\pi\gamma \, dy$ and lift distance y, and evaluating the integral to $50\pi\gamma = 490{,}000\pi \ \mathrm{J}$.

References & Further Reading

- Spivak, M. (2008). *Calculus* (4th ed.), Chapter 18 ("The Logarithm and Exponential Functions") & Chapter 19; for higher-dimensional volumes, Fubini's theorem, and surface integrals over manifolds, see Spivak (1965), *Calculus on Manifolds*, Chapters 3–5. Publish or Perish.
- Apostol, T. M. (1967). *Calculus, Volume 1: One-Variable Calculus, with an Introduction to Linear Algebra* (2nd ed.), Chapter 2 ("Some Applications of Integration") and Chapter 14 (arc length and rectifiable curves). Wiley.
- Stewart, J. (2015). *Calculus: Early Transcendentals* (8th ed.), Chapter 6 ("Applications of Integration") and Chapter 8 ("Further Applications of Integration", hydrostatic pressure and centroids). Cengage.

SECTION DETAIL

Integration problem set

This problem set tests your mastery of integral calculus across Chapter 2. Problems span foundational Darboux sum analysis, partition proofs of integrability, the Leibniz-Newton Fundamental Theorem with dynamic boundaries, and geometric area decomposition.

Problem 2.P.1: Integrability of Step Functions

Proof application

Let $f : [a, b] \to \mathbb{R}$ be a step function; that is, there exists a partition $P^* = \{x_0, x_1, \ldots, x_k\}$ of $[a, b]$ such that on each open subinterval (x_{i-1}, x_i), $f(x) = c_i$ for some constant $c_i \in \mathbb{R}$. At the partition points x_i, $f(x_i)$ may take arbitrary values in $\mathbb{R}$.

Prove from the partition integrability criterion (Definition 2.1.2) that f is Riemann integrable on $[a, b]$, and that its integral equals $\sum_{i=1}^{k} c_i(x_i - x_{i-1})$.

Refine P^* by placing narrow subintervals of width δ around each boundary point x_i where f differs from the constants c_i. Since there are only $k + 1$ boundary points and f is bounded, the total variation contributed by the boundary spikes can be bounded by $2(k + 1)M\delta$.

Let $M = \sup_{x \in [a,b]} |f(x)|$. Since f takes only finitely many values, $M < \infty$.

Given $\varepsilon > 0$, choose $\delta > 0$ such that $2(k+1) \cdot 2M \cdot \delta < \varepsilon$. Construct a refined partition P by placing symmetric intervals $[x_i - \delta, x_i + \delta]$ around each interior partition point x_i (and $[a, a+\delta]$, $[b-\delta, b]$ at endpoints).

On subintervals where f is constant c_i, the supremum equals the infimum, so $M_j - m_j = 0$. The only subintervals contributing to $U(f,P) - L(f,P)$ are the intervals enclosing the partition points x_i. There are at most $2k$ such intervals, each of width at most 2δ, on which $M_j - m_j \leq 2M$.

Therefore:

$$U(f,P) - L(f,P) \leq \sum_{\text{boundary } j} (M_j - m_j)\Delta x_j \leq 2M \cdot 2(k+1)\delta < \varepsilon.$$

By Definition 2.1.2, f is Riemann integrable on $[a,b]$. Furthermore, as $\delta \to 0$, the sum of rectangles on the constant intervals converges to $\sum_{i=1}^{k} c_i(x_i - x_{i-1})$.

Problem 2.P.2: Upper and Lower Darboux Integrals of a Linearly Modulated Dirichlet Function

Proof transfer

Define $f : [0,2] \to \mathbb{R}$ by:

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \cap [0,2] \\ -x & \text{if } x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,2]. \end{cases}$$

Compute the exact upper Darboux integral $\int_0^2 f(x)\,dx$ and lower Darboux integral $\int_0^2 f(x)\,dx$, and deduce whether f is Riemann integrable on $[0,2]$.

On each subinterval $[x_{i-1}, x_i] \subseteq [0,2]$, use the density of $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$ to show that $M_i = x_i$ and $m_i = -x_i$. Compare with the upper and lower sums of $g(x) = x$.

Let $P = \{x_0, x_1, \ldots, x_n\}$ be any partition of $[0,2]$. On each subinterval $[x_{i-1}, x_i]$ with $0 \le x_{i-1} < x_i \le 2$:

1. By density of $\mathbb{Q}$, rationals in $[x_{i-1}, x_i]$ come arbitrarily close to the right endpoint x_i, where $f(r) = r \le x_i$. Since $f(x) \le x \le x_i$ for all $x \in [x_{i-1}, x_i]$, we have $M_i = \sup_{x\in[x_{i-1},x_i]} f(x) = x_i$.
2. By density of $\mathbb{R}\setminus\mathbb{Q}$, irrationals in $[x_{i-1}, x_i]$ come arbitrarily close to x_i, where $f(s) = -s \ge -x_i$. Hence $m_i = \inf_{x\in[x_{i-1},x_i]} f(x) = -x_i$.

Therefore, for every partition P:

$$U(f,P) = \sum_{i=1}^{n} x_i \Delta x_i = U(g,P), \qquad L(f,P) = -\sum_{i=1}^{n} x_i \Delta x_i = -U(g,P),$$

where $g(x) = x$ is continuous and integrable on $[0,2]$ with $\int_0^2 x\,dx = \frac{2^2}{2} = 2$. Taking the infimum of $U(f,P)$ and supremum of $L(f,P)$ over all partitions P:

$$\int_0^2 f(x)\,dx = \inf_P U(g,P) = \int_0^2 x\,dx = 2,$$

$$\int_0^2 f(x)\,dx = \sup_P \big(-U(g,P)\big) = -\inf_P U(g,P) = -2.$$

Because $\int_0^2 f(x)\,dx = -2 \;/=\; 2 = \int_0^2 f(x)\,dx$, f is not Riemann integrable on $[0,2]$.

Problem 2.P.3: The Oscillation Criterion for Integrability

Proof transfer

For a bounded function $f : [a, b] \to \mathbb{R}$ and any subinterval $J \subseteq [a, b]$, define the oscillation of f on J by $\omega(f, J) = \sup_{x \in J} f(x) - \inf_{x \in J} f(x) = \sup_{x,y \in J} |f(x) - f(y)|$.

Prove that f is Riemann integrable on $[a, b]$ if and only if for every $\varepsilon > 0$, there exists a partition $P = \{x_0, x_1, \ldots, x_n\}$ such that:

$$\sum_{i=1}^{n} \omega(f, [x_{i-1}, x_i])\Delta x_i < \varepsilon.$$

Rewrite the definition of $U(f, P) - L(f, P)$ by grouping terms with the common factor Δx_i.

By definition of the upper and lower Darboux sums:

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} M_i \Delta x_i - \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i.$$

Notice that $M_i - m_i = \sup_{x \in [x_{i-1}, x_i]} f(x) - \inf_{x \in [x_{i-1}, x_i]} f(x) = \omega(f, [x_{i-1}, x_i])$.

Therefore, the sum $\sum_{i=1}^{n} \omega(f, [x_{i-1}, x_i])\Delta x_i$ is identically equal to $U(f,P) - L(f,P)$. The claim follows immediately from the Darboux integrability criterion (Definition 2.1.2), which states that f is Riemann integrable on $[a,b]$ if and only if for every $\varepsilon > 0$, there exists a partition P satisfying $U(f,P) - L(f,P) < \varepsilon$.

★ Problem 2.P.4: Integrability of Thomae's Ruler Function

Proof synthesis

Thomae's function $T : [0,1] \to \mathbb{R}$ is defined by:

$$T(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \cap (0,1] \text{ in lowest terms } (p, q \in \mathbb{N}, \gcd(p,q) = 1) \\ 1 & \text{if } x = 0 \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{Q}. \end{cases}$$

Prove that T is Riemann integrable on $[0,1]$ and that $\int_0^1 T(x)\,dx = 0$, despite being discontinuous at every rational number in $[0,1]$.

Notice that for any $N \in \mathbb{N}$, there are only finitely many fractions in $[0, 1]$ with denominator $q \leq N$. Everywhere else, $0 \leq T(x) < 1/N$.

First, for every partition P, every subinterval contains irrationals where $T(x) = 0$. Since $T(x) \geq 0, m_i = \inf_{[x_{i-1}, x_i]} T(x) = 0$, so $L(T, P) = 0$ for all P. Thus, $L(T) = \sup_P L(T, P) = 0$.

To prove integrability, we must show that $\inf_P U(T, P) = 0$. Let $\varepsilon > 0$. Choose an integer $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$. The set of points $S = \{x \in [0, 1] : T(x) \geq 1/N\}$ consists solely of rationals with denominator $q \leq N$. The cardinality $|S| = K$ is finite.

Choose $\delta > 0$ such that $2K\delta < \frac{\varepsilon}{2}$. Construct a partition P of $[0, 1]$ having maximum subinterval width $\Delta x_i < \delta$. Partition the index set $\{1, \ldots, n\}$ into two disjoint sets:

- I_{d}^{ba} : indices where $[x_{i-1}, x_i] \cap S \neq \emptyset$.
- $I_{\text{d}}^{\text{goo}}$: indices where $[x_{i-1}, x_i] \cap S = \emptyset$.

Each point in S can belong to at most 2 adjacent subintervals, so $|I_{\text{d}}^{\text{ba}}| \leq 2K$. On these intervals, $M_i \leq 1$. Thus:

$$\sum_{i \in I_{\text{bad}}} M_i \Delta x_i \leq 1 \cdot \sum_{i \in I_{\text{bad}}} \Delta x_i < 1 \cdot (2K\delta) < \frac{\varepsilon}{2}.$$

On good intervals, $T(x) < 1/N$ everywhere, so $M_i \leq 1/N < \varepsilon/2$. Thus:

$$\sum_{i \in I_{\text{good}}} M_i \Delta x_i \leq \frac{1}{N} \sum_{i \in I_{\text{good}}} \Delta x_i \leq \frac{\varepsilon}{2} \cdot 1 = \frac{\varepsilon}{2}.$$

Combining both sums:

$$U(T, P) = \sum_{i \in I_{\text{bad}}} M_i \Delta x_i + \sum_{i \in I_{\text{good}}} M_i \Delta x_i < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $U(T, P) - L(T, P) < \varepsilon$, T is Riemann integrable by Definition 2.1.2, and $\int_0^1 T(x)\, dx = 0$.

Problem 2.P.5: Stationary Points of a Variable-Limit Gaussian Integral

Construction transfer

Define the function $G : [0, \pi] \to \mathbb{R}$ by:

$$G(x) = \int_{\cos x}^{\sin x} e^{-t^2}\, dt.$$

1. Use the Fundamental Theorem of Calculus (Theorem 2.2.1) and the Chain Rule (Theorem 1.2.4) to compute $G'(x)$.
2. Find all critical points $x \in (0, \pi)$ where $G'(x) = 0$, and determine the exact maximum value of $G(x)$ on $[0, \pi]$.

Split the integral at 0 as $G(x) = \int_0^{\sin x} e^{-t^2}\, dt - \int_0^{\cos x} e^{-t^2}\, dt$, differentiate using the Chain Rule, and observe when $\cos x \cdot e^{-\sin^2 x} + \sin x \cdot e^{-\cos^2 x} = 0$ or when the upper and lower limits coincide.

1. Derivative via FTC Part 1 and Chain Rule: Let $F(u) = \int_0^u e^{-t^2}\,dt$. Since $f(t) = e^{-t^2}$ is continuous everywhere, Theorem 2.2.1 gives $F'(u) = e^{-u^2}$. Writing $G(x) = F(\sin x) - F(\cos x)$ and applying Theorem 1.2.4:

$$G'(x) = F'(\sin x)\cos x - F'(\cos x)(-\sin x) = \cos(x)\,e^{-\sin^2 x} + \sin(x)\,e^{-\cos^2 x}$$

2. Critical Points on $(0, \pi)$: On $(0, \pi/2]$, both $\sin x > 0$ and $\cos x \geq 0$, so $G'(x) > 0$. On $(\pi/2, \pi)$, $\sin x > 0$ and $\cos x < 0$. Notice that at $x = \frac{3\pi}{4}$, we have $\sin(3\pi/4) = \frac{\sqrt{2}}{2}$ and $\cos(3\pi/4) = -\frac{\sqrt{2}}{2}$, so $\sin^2 x = \cos^2 x = \frac{1}{2}$ and:

$$G'\left(\frac{3\pi}{4}\right) = \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) e^{-1/2} = 0.$$

For $x \in (0, 3\pi/4)$, $G'(x) > 0$; for $x \in (3\pi/4, \pi)$, $G'(x) < 0$. Thus $x = \frac{3\pi}{4}$ is the unique critical point and global maximum on $[0, \pi]$, with value $G(3\pi/4) = \int_{\sqrt{2}/2}^{\sqrt{2}/2} e^{-t^2}\,dt$.

Problem 2.P.6: Accumulation of the Floor Step Function

Proof transfer

Let $f : [0, 3] \to \mathbb{R}$ be the floor function $f(t) = \lfloor t \rfloor$, and define the accumulation function $F(x) = \int_0^x \lfloor t \rfloor\,dt$ for $x \in [0, 3]$.

1. Derive an explicit piecewise formula for $F(x)$ on $[0, 3]$.
2. Prove that F is continuous on the entire interval $[0, 3]$, but fails to be differentiable at the interior jump points $x = 1$ and $x = 2$. Explain why this behavior is consistent with Theorem 2.2.1.

On $[0,1)$, $\lfloor t \rfloor = 0$; on $[1,2)$, $\lfloor t \rfloor = 1$; on $[2,3)$, $\lfloor t \rfloor = 2$. Accumulate the area subinterval by subinterval and compare one-sided derivatives at $x = 1$ and $x = 2$.

1. Explicit Piecewise Formula:

- For $x \in [0,1]$: $F(x) = \int_0^x 0 \, dt = 0$.
- For $x \in (1,2]$: $F(x) = \int_0^1 0 \, dt + \int_1^x 1 \, dt = x - 1$.
- For $x \in (2,3]$: $F(x) = 0 + \int_1^2 1 \, dt + \int_2^x 2 \, dt = 1 + 2(x-2) = 2x - 3$.

2. Continuity and Non-Differentiability at Jump Points: At $x = 1$: $\lim_{x \to 1^-} F(x) = 0 = \lim_{x \to 1^+} (x-1) = F(1)$, and at $x = 2$: $\lim_{x \to 2^-} (x-1) = 1 = \lim_{x \to 2^+} (2x-3) = F(2)$. Thus F is continuous on $[0,3]$. However, at $x = 1$, the left-hand derivative is $F'_-(1) = 0$ while the right-hand derivative is $F'_+(1) = 1$. At $x = 2$, $F'_-(2) = 1$ while $F'_+(2) = 2$. Thus $F'(1)$ and $F'(2)$ do not exist. This is consistent with Theorem 2.2.1: the accumulation function of any Riemann integrable function is continuous, while differentiability $F'(x) = f(x)$ is guaranteed at points x where f is continuous (here, on $(0,1) \cup (1,2) \cup (2,3)$).

Problem 2.P.7: Critical Analysis of an Accumulation Function

Description transfer

Let $F : [0,4] \to \mathbb{R}$ be defined by $F(x) = \int_0^x \frac{t-2}{t^2+1} \, dt$.

1. Find all critical points of F on $(0, 4)$ and classify each as a local minimum, local maximum, or inflection point.
2. Determine the intervals on which F is concave upward and concave downward.
3. Identify the location of the absolute minimum of F on $[0, 4]$.

By FTC Part 1, $F'(x) = \frac{x-2}{x^2+1}$. Find where $F'(x) = 0$ and check the sign of $F''(x)$.

1. Critical Points: The integrand $f(t) = \frac{t-2}{t^2+1}$ is continuous on $[0, 4]$. By Theorem 2.2.1:

$$F'(x) = \frac{x-2}{x^2+1}.$$

Setting $F'(x) = 0$ gives $x - 2 = 0 \implies x = 2$. For $x \in (0, 2)$, $F'(x) < 0$ (F is strictly decreasing). For $x \in (2, 4)$, $F'(x) > 0$ (F is strictly increasing). By the First Derivative Test, $x = 2$ is a strict local minimum.

2. Concavity and Inflection Points: Differentiating $F'(x)$ using the quotient rule:

$$F''(x) = \frac{1 \cdot (x^2+1) - (x-2)(2x)}{(x^2+1)^2} = \frac{x^2 + 1 - 2x^2 + 4x}{(x^2+1)^2} = \frac{-x^2 + 4x + 1}{(x^2+1)^2}.$$

Setting $F''(x) = 0$ yields $-x^2 + 4x + 1 = 0 \implies x = \frac{-4 \pm \sqrt{16+4}}{-2} = 2 \pm \sqrt{5}$. Since $2 + \sqrt{5} \approx 4.236 \notin [0, 4]$ and $2 - \sqrt{5} \approx -0.236 \notin [0, 4]$, the numerator $-x^2 + 4x + 1$ is strictly positive for all $x \in [0, 4]$ (at $x = 0$, it is $+1$; at $x = 2$, $+5$; at $x = 4$, $+1$). Therefore, $F''(x) > 0$ for all $x \in [0, 4]$, meaning F is concave upward on the entire interval $[0, 4]$.

3. Absolute Minimum: Since F decreases on $[0, 2]$ and increases on $[2, 4]$, the absolute minimum on $[0, 4]$ occurs uniquely at $x = 2$.

Problem 2.P.8: Locating the Mean Value Point for Integrals

Verification application

Consider $f(x) = 3x^2 - 4x + 1$ on the interval $[0, 2]$.

1. Calculate the exact average value $\mu = \frac{1}{2-0} \int_0^2 f(x)\,dx$.
2. Find all values of $c \in (0, 2)$ that satisfy the Mean Value Theorem for Integrals (Theorem 2.1.4), such that $f(c) = \mu$.

Evaluate the integral using FTC Part 2 (Theorem 2.2.2), then solve the quadratic equation $f(c) = \mu$.

1. Definite Integral and Average Value: An antiderivative of $f(x) = 3x^2 - 4x + 1$ is $F(x) = x^3 - 2x^2 + x$. By Theorem 2.2.2:

$$\int_0^2 (3x^2 - 4x + 1)\,dx = F(2) - F(0) = (8 - 8 + 2) - 0 = 2.$$

The average value is:

$$\mu = \frac{1}{2-0}\int_0^2 f(x)\,dx = \frac{1}{2}\cdot 2 = 1.$$

2. Mean Value Point: By Theorem 2.1.4, there exists $c \in (0, 2)$ such that $f(c) = 1$:

$$3c^2 - 4c + 1 = 1 \iff 3c^2 - 4c = 0 \iff c(3c - 4) = 0.$$

The solutions to the quadratic are $c = 0$ and $c = 4/3$. Since the theorem requires $c \in (0, 2)$ strictly in the open interval, the endpoint $c = 0$ is excluded. The unique valid mean value point is $c = \frac{4}{3} \approx 1.333 \in (0, 2)$.

Problem 2.P.9: Area Between Cubic and Linear Curves

Construction application

Compute the total area of the finite regions bounded between the cubic curve $y = x^3 - 3x$ and the line $y = x$.

Find all intersection points by solving $x^3 - 3x = x$. Note that the curves intersect at three points, so the area must be split into two separate subintervals where the top and bottom curves alternate.

1. Intersection Points: Set $x^3 - 3x = x \iff x^3 - 4x = 0 \iff x(x-2)(x+2) = 0$. The intersection abscissae are $x = -2, x = 0$, and $x = 2$.

2. Relative Positions:

- On $[-2, 0]$: test $x = -1$. Cubic: $(-1)^3 - 3(-1) = 2$. Line: -1. Since $2 > -1$, $x^3 - 3x \geq x$.
- On $[0, 2]$: test $x = 1$. Cubic: $1^3 - 3(1) = -2$. Line: 1. Since $1 > -2$, $x \geq x^3 - 3x$.

3. Integration:

$$A = \int_{-2}^{0} ((x^3 - 3x) - x)\,dx + \int_{0}^{2} (x - (x^3 - 3x))\,dx.$$

Notice that the integrand $g(x) = (x^3 - 3x) - x = x^3 - 4x$ is an odd function, so the areas of the two lobes are symmetric:

$$A_1 = \int_{-2}^{0} (x^3 - 4x)\,dx = \left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0} = 0 - (4 - 8) = 4.$$

$$A_2 = \int_{0}^{2} (4x - x^3)\,dx = \left[2x^2 - \frac{x^4}{4}\right]_{0}^{2} = (8 - 4) - 0 = 4.$$

The total enclosed area is $A = A_1 + A_2 = 4 + 4 = 8$.

Problem 2.P.10: Horizontal Versus Vertical Slicing

Construction transfer

Find the area enclosed between the two parabolas $x = y^2 - 4y$ and $x = 2y - y^2$. Set up and evaluate the integral with respect to y, and explain why horizontal slicing is computationally preferable to vertical slicing for this region.

Solve for the y-coordinates of intersection by equating the two formulas for x. Then integrate $(x_{\text{right}} - x_{\text{left}})\,dy$.

1. Intersection Points: Set $y^2 - 4y = 2y - y^2 \iff 2y^2 - 6y = 0 \iff 2y(y - 3) = 0$. The curves intersect at $y = 0$ and $y = 3$.

2. Orientation: For $y \in (0, 3)$, test $y = 1$: $x_1 = 1^2 - 4(1) = -3$ (left curve). $x_2 = 2(1) - 1^2 = 1$ (right curve). Thus $x_{\text{right}} = 2y - y^2$ and $x_{\text{left}} = y^2 - 4y$.

3. Horizontal Slicing:

$$A = \int_0^3 (x_{\text{right}} - x_{\text{left}})\, dy = \int_0^3 ((2y - y^2) - (y^2 - 4y))\, dy = \int_0^3 (6y - 2y^2)$$

Integrating:

$$A = \left[3y^2 - \frac{2}{3}y^3\right]_0^3 = \left(3(9) - \frac{2}{3}(27)\right) - 0 = 27 - 18 = 9.$$

4. Comparison with Vertical Slicing: Solving $x = y^2 - 4y$ for y yields $y = 2 \pm \sqrt{x + 4}$, and solving $x = 2y - y^2$ yields $y = 1 \pm \sqrt{1 - x}$. Integrating with respect to x requires splitting the domain $[-4, 1]$ into three separate piecewise subintervals $[-4, -3]$, $[-3, 0]$, and $[0, 1]$, and integrating radical expressions on each. Horizontal slicing requires only a single, simple polynomial integral.

Problem 2.P.11: Parabolic Tangent Area Invariance

Proof synthesis

Let $f(x) = x^2$. For any positive constant $a > 0$, consider:

- The parabola $y = x^2$.
- The tangent line to the parabola at the point (a, a^2).
- The vertical y-axis ($x = 0$).

Prove that the area of the region bounded by these three curves is always equal to $\frac{1}{3}a^3$, which is precisely one-third of the area under $y = x^2$ over $[0, a]$.

Find the equation of the tangent line at $x = a$ using $f'(a) = 2a$. Then integrate the vertical difference $(x^2 - y_{\text{tangent}})\,dx$ from $x = 0$ to $x = a$.

1. Tangent Line Equation: The derivative of $f(x) = x^2$ is $f'(x) = 2x$, so the slope at $x = a$ is $m = 2a$. The tangent line equation in point-slope form is:

$$y - a^2 = 2a(x - a) \iff y = 2ax - a^2.$$

2. Bounded Region: On the interval $[0, a]$, the parabola lies above its tangent line:

$$x^2 - (2ax - a^2) = x^2 - 2ax + a^2 = (x - a)^2 \geq 0.$$

The region is bounded on the left by $x = 0$, on the right by $x = a$, from above by $y = x^2$, and from below by $y = 2ax - a^2$.

3. Area Integration: Applying vertical slicing:

$$A = \int_0^a (x^2 - (2ax - a^2))\,dx = \int_0^a (x - a)^2\,dx.$$

Using the substitution $u = x - a$ with $du = dx$ (or expanding directly):

$$A = \left[\frac{(x - a)^3}{3}\right]_0^a = \frac{(a - a)^3}{3} - \frac{(0 - a)^3}{3} = 0 - \frac{-a^3}{3} = \frac{1}{3}a^3.$$

Since $\int_0^a x^2\,dx = \left[\frac{x^3}{3}\right]_0^a = \frac{1}{3}a^3$, the tangent region area $\frac{1}{3}a^3$ equals exactly one-third of the enclosing bounding rectangle $a \times a^2 = a^3$, and matches the area under the parabola.

Problem 2.P.12: Darboux Sum Convergence for the Cubic Monomial

Tool / Simulation | application

Consider $f(x) = x^3$ on the interval $[0, 1]$. For each $n \in \mathbb{N}$, let P_n be the uniform partition with n equal subintervals of width $\Delta x = \frac{1}{n}$, with division points $x_i = \frac{i}{n}$ for $i = 0, 1, \ldots, n$.

1. Using the cubic power-sum identity $\sum_{i=1}^{k} i^3 = \left[\frac{k(k+1)}{2}\right]^2$, derive closed-form formulas in terms of n for $L(f, P_n)$ and $U(f, P_n)$.
2. Prove that $U(f, P_n) - L(f, P_n) = \frac{1}{n} \to 0$ as $n \to \infty$.
3. Deduce $\int_0^1 x^3\, dx$ directly from Definition 2.1.2.

COMPUTE UPPER AND LOWER DARBOUX SUMS FOR F(X) = X^3

```
def darboux_x3(n):
  # For strictly increasing f(x) = x^3 on [0, 1] with uniform partition
of width 1/n:
  # Upper sum uses right endpoints x_i = i / n
  u_sum = sum(((i / n) ** 3) * (1 / n) for i in range(1, n + 1))
  # Lower sum uses left endpoints x_{i-1} = (i - 1) / n
  l_sum = sum(((i - 1) / n) ** 3 * (1 / n) for i in range(1, n + 1))
  return (l_sum, u_sum, u_sum - l_sum)
```

Since f(x) = x^3 is strictly increasing on [0, 1], its supremum on [(i-1)/n, i/n] occurs at the right endpoint x_i = i/n, giving M_i = (i / n) ** 3.

Apply $\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$ with $k = n$ for the upper sum and $k = n-1$ for the lower sum.

1. Closed-Form Darboux Sums: Since $f(x) = x^3$ is strictly increasing on $[0,1]$, on each subinterval $[(i-1)/n, i/n]$ we have $m_i = \frac{(i-1)^3}{n^3}$ and $M_i = \frac{i^3}{n^3}$ with $\Delta x_i = \frac{1}{n}$. Using $\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$:

$$U(f, P_n) = \frac{1}{n^4} \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4n^4} = \frac{(n+1)^2}{4n^2},$$

$$L(f, P_n) = \frac{1}{n^4} \sum_{j=1}^{n-1} j^3 = \frac{(n-1)^2 n^2}{4n^4} = \frac{(n-1)^2}{4n^2}.$$

2. Darboux Gap Convergence:

$$U(f, P_n) - L(f, P_n) = \frac{(n+1)^2 - (n-1)^2}{4n^2} = \frac{4n}{4n^2} = \frac{1}{n} \to 0 \quad \text{as } n \to \infty.$$

By Definition 2.1.2, $f(x) = x^3$ is Riemann integrable on $[0,1]$.

3. Value of the Integral: Taking $n \to \infty$:

$$\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} \frac{n^2 + 2n + 1}{4n^2} = \frac{1}{4} = \lim_{n\to\infty} L(f, P_n).$$

Therefore $\int_0^1 x^3 \, dx = \frac{1}{4}$.

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SECTION DETAIL

Integration recap

This recap sheet synthesises the foundational results of integral calculus developed across Chapter 2. It consolidates Darboux partitions, the Fundamental Theorem of Calculus, geometric accumulation, critical failure modes when hypotheses fail, and standard analytical pitfalls.

Definitions and Results

1. Darboux Partitions and Integrability

- **Partitions and Darboux Sums** (Definition 2.1.1): A partition $P = \{x_0, x_1, \ldots, x_n\}$ subdivides $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of length $\Delta x_i = x_i - x_{i-1}$. For bounded f, the lower and upper Darboux sums are $L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i$ and $U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i$, where $m_i = \inf_{[x_{i-1}, x_i]} f$ and $M_i = \sup_{[x_{i-1}, x_i]} f$.
- **Riemann Integrability** (Definition 2.1.2): The lower and upper Darboux integrals are $L(f) = \sup_P L(f, P)$ and $U(f) = \inf_P U(f, P)$. A bounded function f is Riemann integrable on $[a, b]$ if and only if $L(f) = U(f)$. The common value is denoted $\int_a^b f(x)\, dx$.
- **Continuous Functions are Integrable** (Theorem 2.1.3): Every continuous function $f \in C([a, b])$ is Riemann integrable. Proof relies on uniform continuity on compact $[a, b]$ ensuring oscillation $\omega(f, [x_{i-1}, x_i]) < \frac{\varepsilon}{b-a}$ for sufficiently fine partitions.

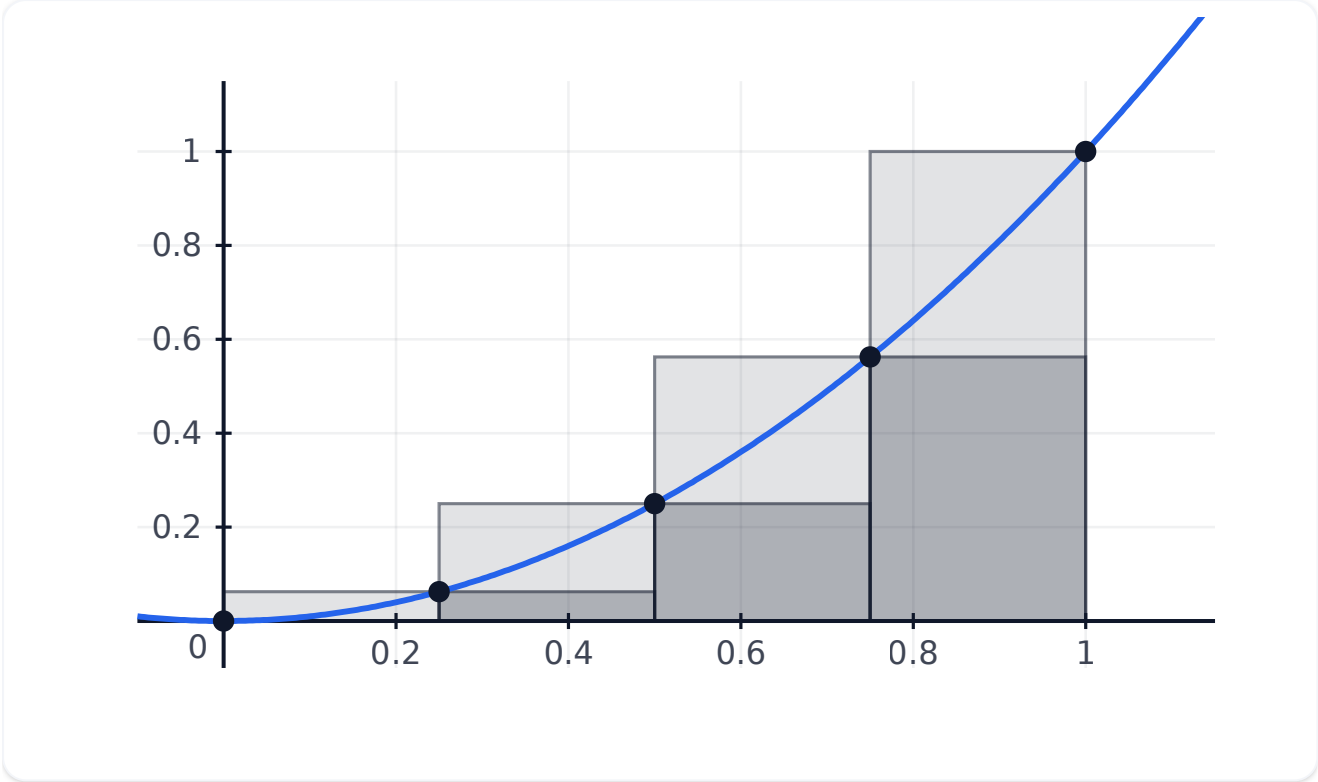


Figure 2.1.1: Darboux sums for $f(x) = x^2$ on $[0, 1]$ with $n = 4$: lower sum $L = \frac{7}{32} = 0.21875$ and upper sum $U = \frac{15}{32} = 0.46875$ sandwich $\int_0^1 x^2 \, dx = \frac{1}{3}$.

2. Properties and the Fundamental Theorem

- **Mean Value Theorem for Integrals** (Theorem 2.1.4): If $f \in C([a, b])$, there exists at least one $c \in [a, b]$ such that $\int_a^b f(x) \, dx = f(c)(b - a)$. Geometrically, the integral equals the area of a rectangle of base $b - a$ and height $f(c)$.
- **Fundamental Theorem of Calculus, Part 1** (Theorem 2.2.1): If f is Riemann integrable on $[a, b]$ and $F(x) = \int_a^x f(t) \, dt$, then F is continuous on $[a, b]$. Furthermore, if f is continuous at $x_0 \in (a, b)$, then F is differentiable at x_0 with $F'(x_0) = f(x_0)$.
- **Fundamental Theorem of Calculus, Part 2** (Theorem 2.2.2): If f is Riemann integrable on $[a, b]$ and F is any antiderivative of f on $[a, b]$ (i.e., $F'(x) = f(x)$ for all $x \in (a, b)$ with $F \in C([a, b])$), then $\int_a^b f(x) \, dx = F(b) - F(a)$.

3. Applications and Geometric Accumulation

- **Volume by Slicing** (Theorem 2.3.1): For a solid whose cross-sectional area perpendicular to the x-axis is given by continuous $A(x)$ for $x \in [a, b]$, the exact volume is the accumulation $V = \int_a^b A(x) \, dx$. For solids of revolution rotated about the x-axis, this yields the disk method $A(x) = \pi [R(x)]^2$ or washer method $A(x) = \pi([R(x)]^2 - [r(x)]^2)$.

- **Arc Length of a Smooth Curve** (Definition 2.3.2): If f' is continuous on $[a, b]$, the total rectilinear arc length of $y = f(x)$ is the limit of polygonal secant lengths: $L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$.

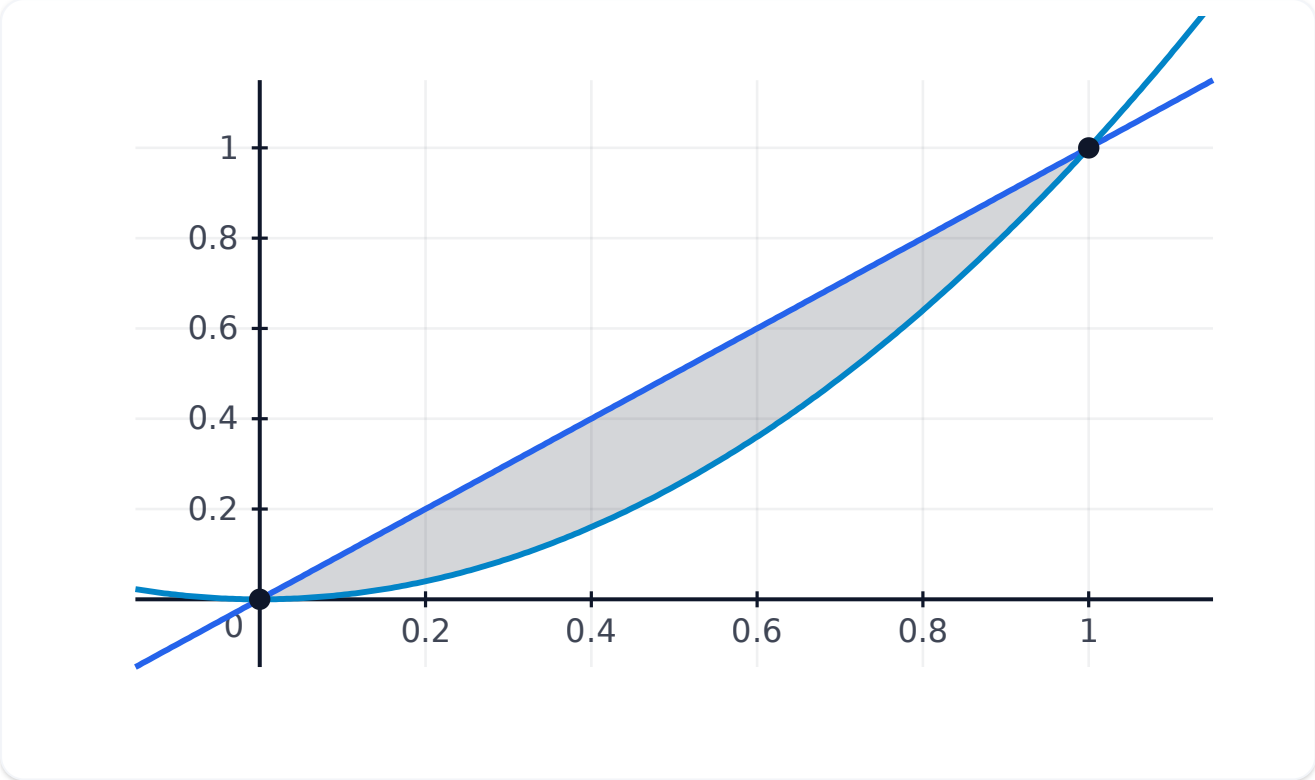


Figure 2.3.1: Area bounded between curves: the region enclosed by $y = x$ and $y = x^2$ on $[0, 1]$ has area $\int_0^1 (x - x^2)\, dx = \frac{1}{6}$.

Method Chooser

Goal / Problem Type	Standard Technique	Key Steps
Prove a function f is integrable	Darboux Oscillation (Definition 2.1.1)	For arbitrary $\varepsilon > 0$, construct a partition P bounding total oscillation $\sum(M_i - m_i)\Delta x_i < \varepsilon$.
Differentiate $\int_{u(x)}^{v(x)} f(t)\,dt$	Theorem 2.2.1 + Leibniz rule	Split at constant c: $\int_c^{v(x)} f - \int_c^{u(x)} f$. Apply chain rule: $v'(x)f(v(x)) - u'(x)f(u(x))$.
Evaluate definite integral $\int_a^b f(x)\,dx$	Theorem 2.2.2	Find antiderivative F such that $F' = f$. Evaluate $F(b) - F(a)$. Check integrand is continuous across $[a, b]$.
Bound unknown integral $\int_a^b f(x)g(x)\,dx$	Monotonicity & MVT (Theorem 2.1.4)	If $m \le f(x) \le M$ and $g \ge 0$, then $m\int g \le \int fg \le M\int g$. Apply Cauchy-Schwarz or MVT for integrals.
Compute volume of solid of revolution	Slicing / Disks (Theorem 2.3.1)	Determine cross-sectional slice area $A(x) = \pi[R(x)]^2$ and integrate along rotational axis $V = \int_a^b A(x)\,dx$.
Compute arc length of graph $y = f(x)$	Rectification (Definition 2.3.2)	Evaluate $L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$ after verifying f' is continuous on $[a, b]$.

Hypothesis Checklist

- **Riemann Integrability (Definition 2.1.2)**
  - *Boundedness is essential:* If $f(x) = 1/x$ for $x \in (0, 1]$ and $f(0) = 0$, then on the first subinterval $[0, x_1]$, $M_1 = \infty$. The upper Darboux sum does not exist as a finite real number.
  - *Countably dense discontinuities:* Dirichlet's function $f = \mathbf{1}_{\mathbb{Q}}$ is bounded on $[0, 1]$, but $U(f, P) = 1$ and $L(f, P) = 0$ for every partition P; $U(f) = 1 \ /= 0 = L(f)$, failing integrability.

- **Fundamental Theorem of Calculus, Part 1 (Theorem 2.2.1)**
  - *Continuity of integrand is required for differentiability of F:* Let $f(x) = \operatorname{sgn}(x)$ on $[-1, 1]$. f is integrable, and $F(x) = \int_1^x f(t)\,dt = |x| - 1$ is continuous. However, F is not differentiable at $x = 0$ because f has a jump discontinuity at 0.
- **Fundamental Theorem of Calculus, Part 2 (Theorem 2.2.2)**
  - *Antiderivative must exist on the entire closed interval $[a, b]$:* Let $f(x) = \frac{1}{x^2}$ on $[-1, 1]$. Naively using the algebraic antiderivative $F(x) = -\frac{1}{x}$ gives $F(1) - F(-1) = -1 - (1) = -2$. But $f(x) > 0$ everywhere, so an integral cannot be negative! The error is that f is unbounded and discontinuous at $x = 0$; F is not defined on $[-1, 1]$.
- **Mean Value Theorem for Integrals (Theorem 2.1.4)**
  - *Continuity of f is required to attain the mean value:* Let $f(x) = 0$ on $[0, 1)$ and $f(1) = 2$. Then $\int_0^1 f(x)\,dx = 0$, giving average value 0, which is attained at $x \in (0, 1)$. But if $f(x) = 1$ on $[0, 1]$ and $f(x) = 3$ on $(1, 2]$, the average is $\frac{1+3}{2} = 2$. Yet $f(x)$ never takes the value 2.

Typical Errors

1. **Blind Antiderivative Evaluation Across Singularities:** Computing $\int_{-1}^{1} \frac{1}{x}\,dx = [\ln|x|]_{-1}^{1} = 0 - 0 = 0$. The integrand is unbounded and discontinuous at $x = 0$; the integral diverges.
2. **Missing Chain Rule in Accumulation Derivatives:** Writing $\frac{d}{dx}\int_0^{x^2} \sin(t)\,dt = \sin(x^2)$ instead of $2x\sin(x^2)$. By the chain rule, $\frac{d}{dx}F(u(x)) = u'(x)F'(u(x))$.
3. **Omitting Absolute Values in Area Integrals:** Integrating $\int_a^b (f(x) - g(x))\,dx$ without verifying which curve is on top. If curves cross at $c \in (a, b)$, the positive and negative lobes partially cancel, underestimating the geometric area.
4. **Confusing Darboux Sums with Darboux Integrals:** $U(f, P)$ depends on a specific partition P. The upper integral $U(f) = \inf_P U(f, P)$ is a unique intrinsic real number.
5. **Swapping Limits Without Sign Inversion:** Forgetting that reversing integration limits inverts the sign: $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

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Approximation

SECTION DETAIL

Taylor Series

Many continuous functions (such as $\sin(x)$, e^x, or $\ln(x)$) cannot be calculated using simple algebraic operations. Power series, and specifically Taylor series, allow us to represent these transcendental functions as "infinite polynomials," which are easy to compute, differentiate, and integrate.

Power Series and Convergence

Before defining Taylor series, we must understand the behavior of power series.

DEFINITION 3.1.1 **Power Series**

A **power series** centered at a is an infinite series of the form:

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

where x is a variable and c_n are the coefficients.

Every power series converges at its center $x = a$. The set of all x for which the series converges is an interval, governed by the radius of convergence.

THEOREM 3.1.2 **Radius of Convergence**

For any power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, exactly one of the following holds:

1. The series converges only at $x = a$ (Radius of convergence $R = 0$).
2. The series converges absolutely for all $x \in \mathbb{R}$ (Radius of convergence $R = \infty$).
3. There exists a real number $R > 0$ such that the series converges absolutely for $|x-a| < R$ and diverges for $|x-a| > R$.

The value R is the **radius of convergence**, and can be calculated using the Root or Ratio test:

$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| \quad \text{or} \quad \frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|c_n|}.$$

REMARK

Although $f(x) = \frac{1}{1+x^2}$ is smooth (C^∞) on the entire real line $\mathbb{R}$, its Maclaurin series $\sum_{n=0}^{\infty}(-1)^n x^{2n}$ has radius of convergence $R = 1$ because its complex extension $f(z) = \frac{1}{1+z^2}$ has singularities at $z = \pm i$ at distance 1 from the origin.

Taylor and Maclaurin Series

If a function can be represented by a power series, the coefficients are uniquely determined by the function's derivatives.

DEFINITION 3.1.3 **Taylor and Maclaurin Series**

Let f be a function with derivatives of all orders on an interval containing a. The **Taylor series** of f centered at a is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

If $a = 0$, the series is called the **Maclaurin series** of f.

Taylor's Theorem with Remainder

In practice, we can only compute finite sums. Taylor's Theorem provides a formula for the error (remainder) when approximating a function with a Taylor polynomial of degree k.

THEOREM 3.1.4 **Taylor's Theorem**

Let f have $k+1$ derivatives on an open interval I containing a. For any $x \in I$, there exists a number c between a and x such that:

$$f(x) = P_k(x) + R_k(x),$$

where $P_k(x)$ is the k-th degree Taylor polynomial:

$$P_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

and $R_k(x)$ is the remainder (error) term, expressed in the **Lagrange form**:

$$R_k(x) = \frac{f^{(k+1)}(c)}{(k+1)!}(x-a)^{k+1}.$$

> ***Pause & Predict: Does $f \in C^\infty$ guarantee convergence to $f(x)$?*** *Suppose a function $f : \mathbb{R} \to \mathbb{R}$ has continuous derivatives of all orders ($f \in C^\infty(\mathbb{R})$), and its Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ converges for all $x \in \mathbb{R}$ (radius of convergence $R = \infty$). Does the sum of the series necessarily equal $f(x)$ for all x? Commit your intuition before opening the analysis below.*

Recall the distinction between smooth (C^∞) and real-analytic (C^ω) functions. Does vanishing derivatives at zero imply the function itself is zero everywhere?

Analysis: No! Having infinitely many derivatives does *not* imply that the Taylor series reconstructs the original function away from the expansion point. A function is called **real-analytic** (C^ω) only if its Taylor series actually converges to $f(x)$ in a neighborhood of a. There exist infinitely differentiable functions (C^∞) that are *not* analytic: their remainder term $R_k(x)$ fails to vanish as $k \to \infty$. We analyze this profound counterexample below.

We fix $x \neq a$. Let M be the unique real number satisfying:

$$f(x) = P_k(x) + M(x-a)^{k+1}.$$

We want to prove that $M = \frac{f^{(k+1)}(c)}{(k+1)!}$ for some c between a and x. Define an auxiliary function $g(t)$ for t in the interval:

$$g(t) = f(t) - P_k(t) - M(t-a)^{k+1}.$$

1. Since f has $k+1$ derivatives, g is continuous and differentiable.
2. Evaluate g at $t = a$:
   - $g(a) = f(a) - P_k(a) - M(0) = f(a) - f(a) = 0$.
   - By definition of $P_k(t)$, we also have $g'(a) = g''(a) = \cdots = g^{(k)}(a) = 0$.
3. Evaluate g at $t = x$:
   - $g(x) = f(x) - P_k(x) - M(x-a)^{k+1} = 0$, by our choice of M.

Since $g(a) = g(x) = 0$, by Rolle's Theorem, there exists c_1 between a and x such that $g'(c_1) = 0$. Now, since $g'(a) = 0$ and $g'(c_1) = 0$, by Rolle's Theorem again, there exists c_2 between a and c_1 such that $g''(c_2) = 0$. Repeating this process $k+1$ times, we find that there exists $c = c_{k+1}$ between a and x such that:

$$g^{(k+1)}(c) = 0.$$

Differentiating $g(t)$ exactly $k+1$ times:

- The $(k+1)$-th derivative of $f(t)$ is $f^{(k+1)}(t)$.
- The $(k+1)$-th derivative of the polynomial $P_k(t)$ of degree k is 0.
- The $(k+1)$-th derivative of $M(t-a)^{k+1}$ is $M \cdot (k+1)!$.

Thus:

$$g^{(k+1)}(t) = f^{(k+1)}(t) - 0 - M(k+1)!$$

Evaluating at $t = c$:

$$0 = f^{(k+1)}(c) - M(k+1)! \implies M = \frac{f^{(k+1)}(c)}{(k+1)!}.$$

Substituting M back into our original equation yields the Lagrange remainder.

■

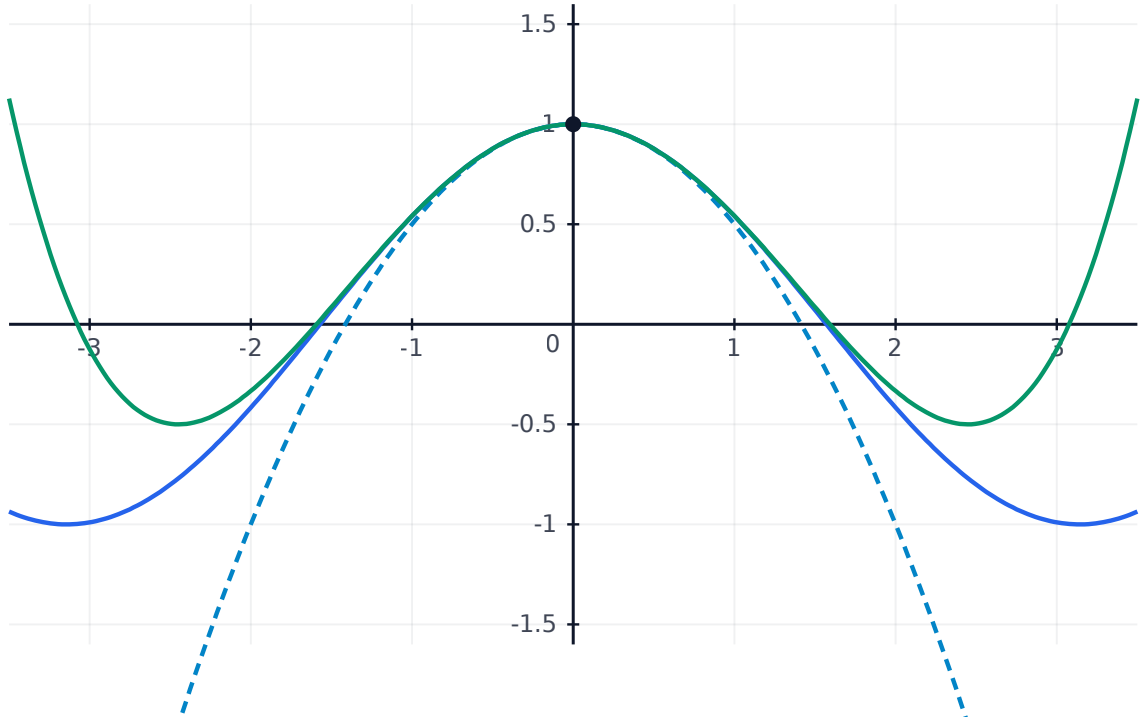


Figure 3.1.1: Maclaurin polynomials $P_2(x) = 1 - \frac{x^2}{2}$ and $P_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$ approximating $\cos(x)$ around $a = 0$.

Let us write a Python script to plot the Maclaurin polynomials of $\cos(x)$ and analyze how adding terms expands the interval where the approximation remains accurate.

INTERACTIVE LAB

```
import math
import numpy as np
import matplotlib.pyplot as plt

def cos_approx(x, order):
  # P_n(x) for cos(x) around a=0
  val = 0.0
  for i in range(0, order + 1, 2):
      sign = (-1) ** (i // 2)
      term = (x ** i) / math.factorial(i)
      val += sign * term
  return val

x = np.linspace(-2 * np.pi, 2 * np.pi, 200)

plt.plot(x, np.cos(x), label='cos(x)', color='black', linewidth=2)
plt.plot(x, cos_approx(x, 2), '--', label='Order 2 (quadratic)', color='r')
plt.plot(x, cos_approx(x, 4), '--', label='Order 4', color='g')
plt.plot(x, cos_approx(x, 8), '--', label='Order 8', color='b')
plt.ylim(-2, 2)
plt.legend()
plt.title("Maclaurin Approximations of cos(x)")
plt.grid(True)
plt.show()

# Compute error at x = np.pi
exact = np.cos(np.pi)
print(f"Exact cos(pi): {exact}")
print(f"Error (order 2): {abs(cos_approx(np.pi, 2) - exact):.6f}")
print(f"Error (order 8): {abs(cos_approx(np.pi, 8) - exact):.6f}")
```

Worked Trace: Rigorous Lagrange Error Bound for e^x

Suppose we approximate $e = e^1$ using the Maclaurin polynomial $P_k(1) = \sum_{n=0}^{k} \frac{1}{n!}$, and we require the approximation error to be strictly less than 10^{-6}.

1. **Formulate the Lagrange Remainder**: By Taylor's Theorem, for $f(x) = e^x$, the $(k+1)$-th derivative is $f^{(k+1)}(x) = e^x$. The remainder at $x = 1$ is:

$$R_k(1) = \frac{e^c}{(k+1)!}(1-0)^{k+1} = \frac{e^c}{(k+1)!}, \quad \text{for some } c \in (0,1).$$

2. **Establish a Rigorous Upper Bound**: Because e^x is strictly increasing, $c < 1 \implies e^c < e^1 < 3$ (using the elementary bound $e < 3$). Therefore:

$$|R_k(1)| < \frac{3}{(k+1)!}.$$

3. **Determine the Required Degree k**: We require:

$$\frac{3}{(k+1)!} < 10^{-6} \iff (k+1)! > 3 \times 10^6 = 3{,}000{,}000.$$

4. **Factorial Growth Trace**:

   - For $k = 8$: $(8+1)! = 9! = 362{,}880 \leq 3{,}000{,}000$ (insufficient).
   - For $k = 9$: $(9+1)! = 10! = 3{,}628{,}800 > 3{,}000{,}000$ (condition satisfied).

5. **Conclusion**: A 9th-degree Taylor polynomial $P_9(1) = \sum_{n=0}^{9} \frac{1}{n!}$ guarantees an error strictly bounded by $\frac{3}{3{,}628{,}800} \approx 8.27 \times 10^{-7} < 10^{-6}$ without needing to know the true value of e in advance.

Counterexample & Failure Mode: Cauchy's Smooth Non-Analytic Function

Does $f \in C^\infty(\mathbb{R})$ imply that the Taylor series converges to $f(x)$? Consider Augustin-Louis Cauchy's classical function:

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

1. **Derivatives at the Origin**: For $x \neq 0$, the n-th derivative has the form $f^{(n)}(x) = P_n(1/x)e^{-1/x^2}$, where P_n is a polynomial. Using the substitution $u = 1/x$, as $x \to 0$ ($u \to \pm\infty$):

$$\lim_{x \to 0} \frac{e^{-1/x^2}}{x^k} = \lim_{u \to \infty} \frac{u^k}{e^{u^2}} = 0 \quad \text{for every integer } k \geq 0.$$

   By induction and the definition of the derivative, $f^{(n)}(0) = 0$ for every $n = 0, 1, 2, \ldots$.

2. **Taylor Series Collapse**: The Maclaurin series of $f(x)$ around $a = 0$ is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} 0 \cdot x^n = 0.$$

   This power series converges for every $x \in \mathbb{R}$ (radius of convergence $R = \infty$), with sum identically zero.

3. **Breakdown Analysis**: For any $x \neq 0, f(x) = e^{-1/x^2} > 0 \neq 0 = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$. Thus, the Taylor series agrees with $f(x)$ **only at the single point** $x = 0$! The function is infinitely differentiable (C^∞) on all of $\mathbb{R}$, yet fails to be analytic (C^ω) at the origin because the remainder term $R_k(x)$ does not converge to zero as $k \to \infty$.

Euler's Formula via Maclaurin Series

By substituting ix into the Maclaurin series of e^z, we can derive the fundamental connection between complex exponentiation and trigonometry.

Let $z = ix$. The Maclaurin series for e^{ix} is:

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \dots$$

Grouping real and imaginary parts:

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right).$$

Recognizing the real part as the Maclaurin series for $\cos(x)$ and the imaginary part as the Maclaurin series for $\sin(x)$, we establish **Euler's Formula**:

$$e^{ix} = \cos(x) + i\sin(x).$$

Worked example. At $x = \pi/2$, the identity gives $e^{i\pi/2} = i$; squaring gives $e^{i\pi} = -1$. Separating the even and odd terms is valid because the exponential power series converges absolutely for every complex z, so its terms may be regrouped. This is an extension of the real-variable series, not a substitute for defining the complex exponential.

SECTION EXERCISES

Deep Practice

Determine radii and intervals of convergence, construct Taylor polynomials, and establish rigorous truncation error bounds via the Lagrange remainder.

Exercise 3.1.1: Radius and Interval of Convergence with Endpoint Analysis

Verification application

Determine the radius of convergence R and the exact interval of convergence (including endpoint behavior) for the power series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n\,2^n}$ (Theorem 3.1.2).

KNOWLEDGE CHECK

Consider the power series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n2^n}$. Evaluate each statement regarding its radius and interval of convergence:

Answer: The radius of convergence is $R = 2$, determined by applying the ratio test limit $|x-3|/2 < 1$.; The series converges absolutely on the open interval $(1, 5)$ centered around the expansion point $x_0 = 3$.

The ratio test gives $\lim |a_{n+1}/a_n| = |x-3|/2$. Convergence requires $|x-3|/2 < 1$, or $|x-3| < 2$; therefore $R = 2$.

After finding $R = 2$ via the Ratio Test, test $x = 1$ (alternating harmonic series) and $x = 5$ (harmonic series) separately.

1. **Ratio test:** Let $a_n = \frac{(x-3)^n}{n\,2^n}$. Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n}{n+1}\cdot\frac{|x-3|}{2} = \frac{|x-3|}{2}.$$

   Convergence holds for $\frac{|x-3|}{2} < 1 \iff |x-3| < 2$, so $R = 2$ and the open interval is $(1, 5)$.
2. **Left endpoint** $x = 1$: The series becomes $\sum_{n=1}^{\infty}\frac{(-2)^n}{n\,2^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, which converges conditionally by the Alternating Series Test.
3. **Right endpoint** $x = 5$: The series becomes $\sum_{n=1}^{\infty}\frac{2^n}{n\,2^n} = \sum_{n=1}^{\infty}\frac{1}{n}$, the harmonic series, which diverges.
4. **Exact interval of convergence:** $[1, 5)$.

Verification criteria. Requires $R = 2$, separate convergence checks at $x = 1$ and $x = 5$, and the half-closed interval $[1, 5)$.

Exercise 3.1.2: Second-Degree Taylor Polynomial and Remainder Bound

Construction application

Construct the quadratic Taylor polynomial $P_2(x)$ of $f(x) = \sqrt{x}$ centered at $a = 4$, use it to approximate $\sqrt{4.2}$, and bound the error $|\sqrt{4.2} - P_2(4.2)|$ using Theorem 3.1.4.

Compute $f(4), f'(4), f''(4)$ for $P_2(x)$, then bound $|f'''(c)| = \frac{3}{8}c^{-5/2}$ on $[4, 4.2]$ using the worst-case endpoint $c = 4$.

1. **Derivatives at $a = 4$:**
   - $f(x) = x^{1/2} \implies f(4) = 2.$
   - $f'(x) = \frac{1}{2}x^{-1/2} \implies f'(4) = \frac{1}{4}.$
   - $f''(x) = -\frac{1}{4}x^{-3/2} \implies f''(4) = -\frac{1}{32}.$
   - $f'''(x) = \frac{3}{8}x^{-5/2}.$

2. **Quadratic Taylor polynomial (Definition 3.1.3):**

$$P_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2.$$

   At $x = 4.2\,(x - 4 = 0.2)$:

$$P_2(4.2) = 2 + \frac{0.2}{4} - \frac{0.04}{64} = 2 + 0.05 - 0.000625 = 2.049375.$$

3. **Lagrange remainder bound (Theorem 3.1.4):** For some $c \in (4, 4.2)$,

$$|R_2(4.2)| = \frac{|f'''(c)|}{3!}(4.2-4)^3 = \frac{3/(8c^{5/2})}{6}(0.2)^3 = \frac{(0.2)^3}{16\,c^{5/2}}.$$

   Since $c > 4, c^{5/2} > 4^{5/2} = 32$, so:

$$|R_2(4.2)| < \frac{0.008}{16 \times 32} = \frac{0.008}{512} = 1.5625 \times 10^{-5}.$$

Verification criteria. Requires $P_2(x)$, numerical evaluation $P_2(4.2) = 2.049375$, and bounding $c^{-5/2}$ at the left endpoint $c = 4$.

Exercise 3.1.3: Uniform Error Bound for the Cubic Maclaurin Approximation of Sine

Proof synthesis

Prove using Theorem 3.1.4 that for every $x \in [-1/2, 1/2]$,

$$\left|\sin x - \left(x - \frac{x^3}{6}\right)\right| \leq \frac{1}{3840}.$$

Note that the even-order derivative $f^{(4)}(0) = \sin(0) = 0$, so $P_3(x) = P_4(x) = x - x^3/6$. Apply the degree-$4$ Lagrange remainder $R_4(x)$ with $|f^{(5)}(c)| = |\cos c| \leq 1$.

1. **Exploit vanishing 4th term:** For $f(x) = \sin x$, we have $f^{(4)}(0) = \sin 0 = 0$, so the 3rd- and 4th-degree Maclaurin polynomials coincide:

$$P_3(x) = P_4(x) = x - \frac{x^3}{6}.$$

2. **Apply Theorem 3.1.4 of order $k = 4$:** There exists c between 0 and x such that:

$$\sin x - P_4(x) = R_4(x) = \frac{f^{(5)}(c)}{5!}x^5 = \frac{\cos c}{120}x^5.$$

3. **Bound on $[-1/2, 1/2]$:** Since $|\cos c| \leq 1$ for all $c \in \mathbb{R}$ and $|x| \leq 1/2$:

$$\left|\sin x - \left(x - \frac{x^3}{6}\right)\right| = |R_4(x)| \leq \frac{1}{120}\left(\frac{1}{2}\right)^5 = \frac{1}{120 \times 32} = \frac{1}{3840}.$$

Verification criteria. Requires recognizing $P_3(x) = P_4(x)$, writing $R_4(x) = \frac{\cos c}{5!}x^5$, and bounding $|\cos c| \leq 1$ and $|x|^5 \leq 1/32$.

Exercise 3.1.4: Implementing the Maclaurin Series of the Exponential Function

Construction application

Complete the Python implementation of the Maclaurin partial sum $\sum_{n=0}^{N-1} \frac{x^n}{n!}$ for e^x:

COMPLETE THE MACLAURIN SERIES FUNCTION FOR E^X

```
def exp_maclaurin(x, terms):
  val = 0.0
  import math
  for n in range(terms):
      # General term is x^n / n!
      val += (x ** n) / math.factorial(n)
  return val
```

The n-th term in the Maclaurin expansion of e^x is x^n / n!.

References & Further Reading

- Spivak, M. (2008). *Calculus* (4th ed.), Chapter 20 (“Approximation by Polynomial Functions”, including Lagrange and integral remainder forms) and Chapters 25–27 (complex power series and why singularities in $\mathbb{C}$ govern real radius of convergence). Publish or Perish.
- Rudin, W. (1976). *Principles of Mathematical Analysis* (3rd ed.), Chapter 5 (Taylor’s Theorem, Theorem 5.15) & Chapter 8 (“Some Special Functions”, power series and Fourier series). McGraw-Hill.
- Apostol, T. M. (1967). *Calculus, Volume 1: One-Variable Calculus, with an Introduction to Linear Algebra* (2nd ed.), Chapter 7 (“Polynomial Approximations to Functions”) and Chapter 11 (“Infinite Series and Improper Integrals”). Wiley.

Course Practice

SECTION DETAIL

Calculus Practice Workshop

Problem Workshop

This workshop generates algorithmic practice problems spanning single-variable calculus: limits and continuity, differential calculus and the Mean Value Theorem, Riemann integration and the Fundamental Theorem of Calculus, and Taylor polynomials with remainder bounds.

Recommended Workflow

1. Derive complete solutions on paper before inspecting hints.
2. Consult progressive hints when reaching an algebraic or conceptual impasse.
3. Verify derivations against the complete worked solution, checking hypothesis conditions and domain restrictions.

INTERACTIVE PRACTICE WORKSHOP

Randomized parameter generation, progressive hints, and worked derivations are available in the interactive web edition. Covered topics:

- Limits & Continuity
- Derivatives & Applications
- Integration & Accumulation
- Series & Approximations

References & Further Reading

- Stephen Abbott (2015). *Understanding Analysis* (2nd ed.), Chapters 2, 4, 6, and 7. Springer.
- Michael Spivak (2008). *Calculus* (4th ed.), Chapters 5, 9, 11, 13, and 20. Publish or Perish.
- Walter Rudin (1976). *Principles of Mathematical Analysis* (3rd ed.), Chapters 4, 5, and 6. McGraw-Hill.